

Ian Stewart (2008) devotes pp. 8-19 and 40-53 to an overview of various ancient and exotic number systems. On p. 138, he introduces us to the hierarchal system comprised of the *natural* numbers, the *integers*, *rationals*, *irrationals* and *real* numbers. (With slight variations in nomenclature — e.g., with *cardinal*, *counting* or *concrete* employed as a synonym for *natural* — we find the same model reflected in Clegg pp. 150-152; Gullberg pp. 5, 70-1, 157; Jourdain p. 21; etc.) Similarly, Gullberg 1997 has a chapter entitled ‘Systems of Numeration’ but its 38 pages are dominated by the ancient and exotic, with our own system of Indo-Arabic numerals covered only in a nuts-and-bolts fashion as he recounts their introduction to the West by Fibonacci in 1202 (page 50). By ‘nuts-and-bolts’ I mean there is no hint in Gullberg of the kind of philosophical issue raised by Clegg above. A similar approach is found here: Joseph 2011: 30-75, 198-206, 338-339, 460-466. In each case, having arrived by a richly detailed historical path at the (terribly named) ‘real number line’ upon which so many impressive edifices and sky castles may be built, it feels as if the the whole waterfront surely was covered.

But what about the moment-before-Peano on which Clegg focuses for a moment? It is nowhere to be found in the standard presentations. At first glance one might think that Dedekind’s essay on ‘The Nature and Meaning of Numbers’ (1963[1888] 31-115) might cover the territory that I say is neglected, but his overarching agenda turns out to be quite the opposite: “With reference to this freeing the elements from every other content (abstraction) we are [now] justified in calling numbers a free creation of the human mind” (68; cf. 31). I.e., the essay is all about securing the cloud castle and guaranteeing that it does *not* touch the ground, so to speak. An analogy: After axiomization, the mathematician focuses on the new turrets and spires that extend the citadel upward, never looking back at the foundational integers, just as a construction worker would be focused on the beam where he walks, 57 stories above the sidewalk, not second-guessing details of the building’s foundation. But there is no law against someone *else* revisiting the foundational level, either to praise it or question it. That’s what we are doing here, as ‘outsiders’ to the profession.

As we try to pursue this question of how real the numbers are, it doesn’t help that Hawking chose *God Created the Integers* as the title of his thousand-page mathematics anthology. The title comes from the following *bon mot* of Kronecker’s: “The dear Lord made the integers; all

else is the work of humans” (my translation of *Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk* on de.wikipedia). Inside the anthology itself, the title’s attribution comes very late (Hawking 2005:892) and with such a perfunctory glance at the historical context that the general reader is still left in the dark as to its purport. Only from other sources, such as Clegg 68, 190-194 or Dauben 66-70, can one learn what Kronecker actually meant: “The mathematics of integers is sufficiently rich to keep us fully occupied. Other types are superfluous to us mortals or simply nonexistent; their pursuit, as by Lindemann and by Cantor, is unseemly and foolish” (my paraphrase).

Clearly, Kronecker was a reactionary, a figure that many would dismiss as having been on the wrong side of history, not part of the club. Yet Hawking is so enamored of his *bon mot* that he uses it willy-nilly as a book title, confident that most potential purchasers will know little or nothing of the real Kronecker. Meanwhile, far from sounding reactionary, Kronecker sounds *to me* like the voice of reason, by the way:

As nothing less than the whole edifice from Eudoxus to Cantor is at stake, little wonder that these views [of Kronecker’s] cause a stir in the mathematical world. ‘Of what use,’ said Kronecker to Lindemann, ‘is your beautiful investigation regarding π ? *Why study such problems, since irrational numbers are non-existent?*’ So back we are once more at a logical scandal such as troubled the Greeks. The Greeks survived and conquered it, and so shall we. At any rate, it is all a sign of the eternal freshness of mathematics.
—Turnbull, 1951 [1929], I:168, italics added. (The allusion is to Lindemann’s 1882 demonstration of π ’s transcendence.)

Most of the issues raised in the present article turn out to be distinctly Kroneckeresque, but that is beside the point here.

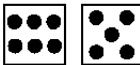
Given all the double-think and shenanigans (such as Hawking’s shell game about God and the Integers as described above), how can we break through to the numeration system itself and clear the air? For that purpose, let’s turn now to a thought experiment about the number eleven. (For this thought experiment, I take my inspiration from Rahman 2014:66.) Think of ‘eleven’, please, then match one of the items in Figure 5 to your mental image of it.

Items [a] through [f] are some of the possible responses to “Think of eleven,” varied by occupation (or preoccupation); psychology; temperament; nationality; and so on. There is no ‘wrong answer’; sadly for us simians, neither is there a right answer, for all of them look equally awkward and irrelevant against a cosmic backdrop.

[a] 11

[b] B

[c] 1011

[d] 

[e] 

[f] ‘eleven’ (or ‘*unsprezece*’ in Romanian, etc.)

The exercise was inspired by “Think of ‘elephant,’ think of ‘fifteen’ ” in Rahman 2014:66 (which in turn might be an allusion to “I can’t hold 15 in my hand” in Clegg 2003:152?)

Figure 5: What is ‘eleven’?

For most of us, the answer will be [a]. A firmware engineer, if working long hours recently, might choose [b], where we show eleven in hexadecimal, or conceivably [c], where we express eleven in binary notation. Similarly, after a busy night at the casino a gambler might plausibly envision [d]. Someone on the autism spectrum, inclined to think visually about everything, might respond with [e], where my intention is to represent his/her mental picture of, say, marbles, conveniently subgrouped. Finally, to complete the list, a purely literary person (or a smart-aleck) might think ‘eleven’ (or *unsprezece* in Romanian). Fine. There are no wrong answers — yet.

Now we turn to the real question: Of the half-dozen choices, which would the Creator (or, if you prefer, an advanced extraterrestrial) point to as *the* number eleven itself? On reflection, we see that none of our answers fits the bill as *the* number eleven itself. The base-10 notion that undergirds ‘11, 12, 13...’ is the legacy of our ancestors who bequeathed us two thumbs and eight fingers, echoed by ten toes. No gift from *der liebe Gott* there, only branch-grabbing happenstance. Nor can ‘B’ lay claim to being the actual eleven, given the abstruse nature and historically late formulation of its hexadecimal base. Ditto for binary. As for marbles in a queue, intuitively one would argue that such is not abstract *enough* to be a serious contender. (Cf. Jourdain pp. 21 and 35 on ‘symbols of number.’) Likewise ‘eleven.’ Candidly, we haven’t a clue

how the genuine number eleven should be represented even here on earth, never mind on a higher plane, or in God's House.

Taking it a step further: In the cosmic eye, is there even such a thing as the counting numbers (1, 2, 3...) by whatever names/symbols/axiomatic scheme? True, we can readily point to *indirect* demonstrations that we are 'thinking correctly,' such as 'making an atomic bomb' or 'navigating to the moon,' but where is the *direct* confirmation that our counting numbers are real, not just an artifact of the culture on monkey-planet? Suppose we cite the composition of the chemical elements, with 1, 2, 3 protons defining hydrogen, helium, lithium...? There I believe we might have our first (and only?) sign of encouragement, reflected back from the universe itself.

Concluding Remarks

Getting real: "But like a diseased financial giant, isn't the Institution of Pi 'too big to fail'?" Point taken. Accordingly, I propose that the scalpel be wielded carefully, as follows: In physics and engineering, the symbol ' π ' should stay since it causes no harm there, in its role as a practical tool. The place where its use should be discouraged is, ironically, within the citadel of Queen Mathematics herself. There, to counteract 20-odd centuries of crypto-mysticism and double-think, π should be replaced, in the vast majority of situations, by '3.14' or '3.1415' or 'the 3.1415-algorithm' or 'the 3.1415 algorithm family.' And similarly for ϕ , $\sqrt{2}$ and e . After all, 3.1415, 1.6180, 1.4142 and 2.7182 all have strong personalities and are quite capable of 'announcing themselves' on the instant, without the aid of religious iconography.

Similarly, one might argue that ' ∞ ' as used in physics (e.g., to define the maximum energy level for an electron, opposite of its ground state energy) is a useful tool. (Note the term 'pure mathematics' in the Nagel/Newman passage quoted in the Prologue.) But elsewhere, there are numerous situations where traditional ' $\lim_{n \rightarrow \infty}$ ' should be replaced by 'algorithm, for Ξ ' or 'algorithm, for \aleph ' (borrowing the Devanagari letter *ka* that occurs in *kalp* 'kalpa' and *anant kaal* 'eternity'). That way, the distinction between an eternally living asymptotic curve and its dead abstract limit on the whiteboard is not smeared about for the simian convenience of 'getting an answer' or 'breaking for lunch.'

In sorting this out, it helps also to make a three-way distinction between *tags*, *symbols* and *names*. The term ‘mileage’ is understood as a useful *tag*, nothing more or less. I.e., no one believes that ‘the mileage on a Honda’ denotes an actual *thing*. The symbol π should be used the same way, and so it is in physics and engineering, where it plays the role of pragmatic *tag*. But in mathematics, the symbol π has become more like the name ‘Santa Claus’ where some who hear the *name* mistake it for evidence of the existence of a personage or a *thing*.

Above I have made engineering out to be one of the ‘good guys’ but in fairness we should note also the following negative role that engineering has played in the story:

Far from pleasing the faculty councils at the École, Cauchy’s *Calcul Infinitésimal* incurred their wrath for being too theoretical and not sufficiently practical. In fact, in late 1823, the Minister of the Interior appointed a commission including Laplace and Poisson to ensure that the instruction in mathematics was attuned to the needs of engineering students. For the rest of the decade the École’s administration continually monitored Cauchy’s lectures to guarantee their suitability for engineering students. —Hawking 2005:641

This provides an important clue as to why certain concepts in the area of limits and asymptotes were gradually degraded in mathematics, so that Cauchy’s ‘converge toward’ was supplanted by ‘converge to’, the flea-hop, etc., as described above. It seems likely that continuing pressure from Engineering departments, not just in France in the 1800s but everywhere for the ensuing two centuries, drove those changes.

Note that many parts of conventional mathematics remain untouched by the ‘assault’ above on some sacred cows. Left standing, as it were, are many cherished parts of the landscape, including the ζ -function (see de Sautoy, *The Music of the Primes*); i (see Nahin); the Mandelbrot set; bifurcation nodes as they relate to Feigenbaum’s 4.669201 (see reference above to Ekeland 1988, Appendix 2); calculus as the celebration of curves (Boyce 2013, Chapters III-IV); and, as discussed already in connection with Ramanujan, the *right-hand* side of the so-called ‘ π equation’ of your choice.

On the long-term prospects for our ‘Queen of the Sciences’ (Newman I:294, quoting Gauss in praise of mathematics). Once upon a time, the word *alchemy* simply meant ‘chemistry’ (from Greek *khēmia*, ‘the art of transmuting metals,’ by way of Arabic *alkīmiyā*). Do present-day chemists look down upon the alchemists of yore? Not exactly. Theirs is accepted as an important protoscientific tradition. Still, some of their specific pursuits are frowned upon, even ridiculed,

e.g., taking quicksilver (Hg) as an elixir (presumably because ‘quick’ means ‘living’). And in forming an overall impression of alchemy today, it is difficult to avoid the words *quaint* and *wrongheaded*, somewhere in the mix. I predict that present-day mathematics, if it does not clean up its act, will suffer a fate similar to that of alchemy. A few centuries hence, it will be abandoned in favor of something else (computer science is a good candidate), and remembered only as an important phase in our simian development, now too quaint to warrant much attention, except by historians.

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