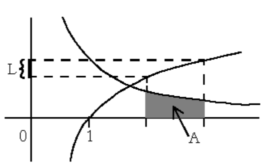
Example	Functionality	Comments
[Let] n = 1, 2...	*assignment* of a value or values to n	Given the stark contrast between *assignment* and *equality*, all programming languages make a clear distinction of '=' vs. '= =' (or ':=' vs. '='), respectively. Meanwhile, how does mathematics get by without such symbology? The one symbol '=' is called upon to handle both in a manner that may be termed 'overloading' with the caveat that this particular flavor of overloading developed in a vague, ad hoc manner over the centuries, not as a deliberate strategy à la *operator overloading* in software engineering. Lacking the operator '= =' and not quite trusting overloaded '=' for the job, the mathematics author often feels constrained to test for equality this way: '$a - b = 0$' (Zill 4-5), thus bloating the syntax with gratuitous operands and operators. See text.
[If] $a = b$ [then...]	*equality test* of a and b (i.e., the 'a = = b' test of programming languages)	
$\lim\limits_{x \to a} f(x) = 1$	*copula* for a limit value	In column 1, the '=' corresponds to 'is' (the copula) in the following sentence: "The limit value of function $f(x)$, as x approaches a, is 1" (after Gullberg 356). To say that $f(x)$ *equals* something (= 1) would be to trash the very notion of an asymptotic process, but even if we interpret '=' to mean 'is' (via additional overloading) a problem remains: The term *limit value* is itself bogus, a sleight-of-hand trick: A function may have a value; a limit is simply a limit; and never the twain should meet; but here they do. Why? See discussion in text.
$\lim\limits_{n \to \infty} S_n = 42$ copula in a *limit-surrogate* relation such as: "[The $\zeta(s)$ series] sums to the limit $\frac{\pi^2}{6}$" (Hawking 2005:822) or: "$\zeta(2)$ [...] = $\frac{\pi^2}{6}$" (Conway/Guy 262) $\sum_{n=1}^{\infty} a_n = \lim\limits_{n \to \infty} \sum_{i=1}^{n} a_i$		The first example is after 'sum to infinity' in Gullberg 270. (See also Devaney 12, 109-112.) Protter/Morrey 48 has this: "[$n \to \infty$ is] used to define a limit at infinity." N.B. the phrases '*to* infinity' and '*at* infinity'; contrast '*for* eternity' as discussed in the text. Comment on 'sums *to* the limit' in Hawking 822: A series has *no* such 'sum'; nor can π be part of *any* 'limit' since π is a process, not a thing. One should say that the series converges *toward* $\pi^2/6$, as it (too) runs *for* eternity. (In such contexts, Cauchy used 'converge toward' [*converga vers*], but over time, his precise language has been supplanted by crude rephrasings or/and inept translations. See text.) The example with Σ is from J. Stewart 729; see comments in text.
$e^{i\pi} = -1$	fudging of a supposed *identity* relation	Here we see the famous 'identity' by which e, i and π seem to have been distilled to celestial heights and there set 'equal to –1'. But it turns out that the –1 in that relation is only a limit. See Conway/Guy 254-256 and remarks in text.
$\pi = 4 \times \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \cdots\right]$ $4 \times \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \cdots\right]$ ¥ 3.1415... And to avoid the lie of '√2 = ...' write this: $\prod_{k=0}^{\infty} \left(1 + \frac{1}{4k+1}\right)\left(1 - \frac{1}{4k+3}\right)$ ¥ 1.414213... Similarly, to avoid the lie of 'ϕ = ...' one should write this: $\frac{(1+\sqrt{5})}{2}$ ¥ 1.6180... And rather than set something 'equal' to e, one should write: $\lim\limits_{x \to 0+} (1+x)^{1/x}$ ¥ 2.7182...		The first equation says, in effect, "The mathematical 'constant π' will find, at the end of the rainbow, its 'true value,' by means of an infinite series that we pretend is kind-of-asymptotic-like, although we know better, as we thus trivialize the sacred concept of limits." To avoid such abuse of the equals sign and the concept of limits, we need a brand-new symbol, say ¥, meaning '... is a process that yields...', as demonstrated at the left with '...¥ 3.1415...', etc. where four accustomed lies have been turned into honest statements. The fourth example alludes to Protter/Morrey 121; sources for the others are in the text. Counterexample: In sources such as Gullberg 778 and Hawking 822 and 876, one is encouraged to see references to 'the function $\zeta(s)$' of Euler and Riemann, where the library of 'constants' comprised of nonexistent numbers might otherwise have been 'enriched' by 'the ζ number 1.644' (≈ $\pi^2/6$).
 $\int_a^b f(x)\,dx = \mathrm{F}(b) - \mathrm{F}(a)$ Smack in the middle of the Fundamental Theorem of Calculus, we find what? An *equals* sign. But how can an area A equal a distance L?		Once again, we find ourselves emerged in double-think: On the one hand, Dimensional Analysis is a foundation stone of the sciences; on the other hand, the very thing that makes calculus valuable is lodged in its treatment of dimensions n and n+1 as interchangeable. Thus, one pretends the units of F(b) – F(a) are m^2, when really they are m. Or, conversely, by sleight-of-hand one implies that the units of an integration are m when really they are m/s, as in Stewart 2012:335. Yes, one has finessed the situation, but at the cost of trivializing calculus, as though it were just a dubious student crib, not a major discovery. (The illustration of 'A = L' is after Boyce 2013:44-46.)

Figure 4: Some of the half-dozen ways that '=' is overloaded in mathematics

In software engineering, where assignment and equality are like oil and water, the use of a single symbol to handle both would lead to rampant bugginess. The tacit blurring of those two concepts in mathematics manifests in other ways: sometimes in vagueness, sometimes in expressions that are precise but strangely formulated. Take the syntax '$a - b = 0$'. In mathematics, this is a species of equality statement or equality test (e.g., in Zill 4-5). In such an equation, it seems that we are concerned with three entities (a, b and 0) as they relate to two operators, but logically, given the purport of the statement, it should contain only *two* entities and a *single* operator (like 'a = = b' in C or Java for example). Thus, one of the more annoying tics of the language of mathematics.

In row 3, we have "The limit value of function $f(x)$, as x approaches a, is 1" (after Gullberg 356). In precalculus, the concept of a limit is developed very slowly (e.g., over a 50-page span in Hungerford 2004:828-877), with care that verges on the quasi-religious. And with that tradition I have no quarrel since the concept *is* subtle and *is* inherently interesting, and practical to boot. But in the calculus curriculum proper, the concept of a limit is often abused. The trouble comes with asymptotic functions that converge *toward* a limit. While Nature is slow and patient, we simians are the opposite, so a 'culture clash' ensues. A function may have a *value*, and a limit is simply a *limit*; why would one jam the two words together as 'limit value'? That argot of the mathematics classroom is a way of trying to disguise or dignify the practice of abandoning an asymptotic process at an arbitrary time (e.g., lunchtime) to perform a flea-hop *from* the function *onto* its limit. The math instructor may keep the distinction clear in his/her head, but the student will surely just take things at face value, too busy to notice the shell game until months later, if ever.

In the next row of Figure 4 we have 'sum to infinity' (Gullberg 270) and 'sums to the limit $\pi^2/6$' (Hawking 822). Here I append two related passages, not cited in Figure 4 for lack of space: "[It] was a sheer assumption that such a *process* as 1.4142... [...] has any limit at all" (Jourdain 2013 [1913]:59-61, italics added). "[By the epsilon method] we no longer need to take [1 + ½ + ⅓ +...] all the way *to* infinity to get the *value of the sum* to be 2" (Clegg 2003:126, italics added). We may count the Jourdain passage as a brief moment of sanity flashing by in 1913. Conversely, the Clegg passage is like an x-ray in which all parts of the prevalent double-think anatomy are suddenly lit up together. (In that way, it is akin to the 'smoking gun' passage quoted earlier from de Sautoy 67.)

A final note about sums and limits. What could be more straightforward than the summation symbol, Σ? With subscripted $i=m$ and superscripted n, it indicates "the sum of all [terms] as i goes from m to n" (Gullberg 105). Replace the n by ∞, and we have *the* ultimate math icon, used to good effect on the spines of Newman's anthology, for instance, where the volume numbers are indicated by subscripted i=1,2,3,4. However, for the variant that has ∞ on top (see example in row 4 of Figure 4, after Stewart 729), some authors feel compelled to issue a warning: "It is important to note that the sum of a series is not a sum in the ordinary sense. It is a limit." (Salas/Hille 614.) What the warning means is that we are now back in the realm of quicksand and double-think where an eternal series *converges* '*to*' its limit so that the limit can morph into its 'sum'. In such contexts, Cauchy used 'will converge *toward* [*converga vers*] a limit', but over time, his precise language has been supplanted by crude rephrasings, whereby 'toward' morphs into 'to'. See Cauchy as presented directly (albeit in translation) in Hawking 655, 658; then compare Hawking's paraphrase of Cauchy using a 'modern' idiom, on p. 640.

In the so-*called* identity, $e^{i\pi} = -1$, of Euler we find a Holy Trinity of sorts. But of the three, only i is legitimate, which is ironic since i stands for imaginary. (I call i legitimate for reasons explained in Nahin passim.) Meanwhile, e does not exist and π does not exist (as an object or 'constant'). Moreover, even if those two existed as bona fide numbers, the right-hand side, '= –1', turns out *not* to be an identity relation after all (though widely advertised as such). Rather, –1 is a limit; see Conway/Guy 1996:255. Thus, the symbol '=' has once again been overloaded, this time in the role of 'helping to fudge an identity relation' by lying about the right side of an equation (whose left side is also bogus for the reasons mentioned above, but that is not the main point here). The row in Figure 4 devoted to π (and ¥ = *yields*) contains examples that hark back to the Ramanujan equation shown earlier.

Part Three: Our numeration blind spot

We live in (collective) ignorance of what numbers are. While the lay person would object with indignation to that statement, it would come as no surprise to the mathematician. This harks back to the italicized phrase '*purely logical consequences*' in the passage we quoted from Nagel/Newman in the Prologue. But even if Queen Mathematics has summarily swept 'What numbers are' off the table, there is no law that forbids others from expressing curiosity about the crumbs of that question that remain on the floor. But before confronting the topic directly (with help from Figure 5), let's get a feel for how it seems *almost* to exist, *somewhere* in the vastness of the conventional math universe.

At the very moment of its birth (say 1889), *number* theory simultaneously gives a perfunctory nod to the *numeration* question, and promptly washes its hands of it. (Cf. Joseph p. 35. See also the fleeting mention of 'symbols of number' in Jourdain p. 21.) Consider the following reflection on how/why Peano axiomized numbers, taking the 'natural numbers' (aka 'counting numbers') as his foundation:

> It might seem strange that Peano should need to [develop our 'counting numbers' from a set of propositions] but those familiar numbers we use all the time [...] have to come from somewhere [...] it's easy to think of them as *real things* [...] But in reality these numbers [...] are *just symbols* we use to represent the cardinality of a set. I can't hold 15 in my hand. [Peano in 1889] lets us build those numbers in terms of a series of sets that are almost hauled up by their own bootstraps. —Clegg 2003:152, italics added.

Early on, Peano would have been well aware of (indeed driven by) the fact that our numbers are not 'real things,' but once he entered the realm of axioms, all attention was focused there and the philosophical question would have been spirited away to the far horizon, where it became 'not my job' for the mathematician. Let's look at some representative examples to see what *is* typically covered, just 'inches away' from what would be numeration theory, if such existed.

Ian Stewart (2008) devotes pp. 8-19 and 40-53 to an overview of various ancient and exotic number systems. On p. 138, he introduces us to the hierarchal system comprised of the *natural* numbers, the *integers*, *rationals*, *irrationals* and *real* numbers. (With slight variations in nomenclature — e.g., with *cardinal*, *counting* or *concrete* employed as a synonym for *natural* — we find the same model reflected in Clegg pp. 150-152; Gullberg pp. 5, 70-1, 157; Jourdain p. 21; etc.) Similarly, Gullberg 1997 has a chapter entitled 'Systems of Numeration' but its 38 pages are dominated by the ancient and exotic, with our own system of Indo-Arabic numerals covered only in a nuts-and-bolts fashion as he recounts their introduction to the West by Fibonacci in 1202 (page 50). By 'nuts-and-bolts' I mean there is no hint in Gullberg of the kind of philosophical issue raised by Clegg above. A similar approach is found here: Joseph 2011: 30-75, 198-206, 338-339, 460-466. In each case, having arrived by a richly detailed historical path at the (terribly named) 'real number line' upon which so many impressive edifices and sky castles may be built, it feels as if the the whole waterfront surely was covered.

But what about the moment-before-Peano on which Clegg focuses for a moment? It is nowhere to be found in the standard presentations. At first glance one might think that Dedekind's essay on 'The Nature and Meaning of Numbers' (1963[1888] 31-115) might cover the territory that I say is neglected, but his overarching agenda turns out to be quite the opposite: "With reference to this freeing the elements from every other content (abstraction) we are [now] justified in calling numbers a free creation of the human mind" (68; cf. 31). I.e., the essay is all about securing the cloud castle and guaranteeing that it does *not* touch the ground, so to speak. An analogy: After axiomization, the mathematician focuses on the new turrets and spires that extend the citadel upward, never looking back at the foundational integers, just as a construction worker would be focused on the beam where he walks, 57 stories above the sidewalk, not second-guessing details of the building's foundation. But there is no law against someone *else* revisiting the foundational level, either to praise it or question it. That's what we are doing here, as 'outsiders' to the profession.

As we try to pursue this question of how real the numbers are, it doesn't help that Hawking chose *God Created the Integers* as the title of his thousand-page mathematics anthology. The title comes from the following *bon mot* of Kronecker's: "The dear Lord made the integers; all else is the work of humans” (my translation of *Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk* on de.wikipedia). Inside the anthology itself, the title’s attribution comes very late (Hawking 2005:892) and with such a perfunctory glance at the historical context that the general reader is still left in the dark as to its purport. Only from other sources, such as Clegg 68, 190-194 or Dauben 66-70, can one learn what Kronecker actually meant: “The mathematics of integers is sufficiently rich to keep us fully occupied. Other types are superfluous to us mortals or simply nonexistent; their pursuit, as by Lindemann and by Cantor, is unseemly and foolish” (my paraphrase).

Clearly, Kronecker was a reactionary, a figure that many would dismiss as having been on the wrong side of history, not part of the club. Yet Hawking is so enamored of his *bon mot* that he uses it willy-nilly as a book title, confident that most potential purchasers will know little or nothing of the real Kronecker. Meanwhile, far from sounding reactionary, Kronecker sounds *to me* like the voice of reason, by the way:

> As nothing less than the whole edifice from Eudoxus to Cantor is at stake, little wonder that these views [of Kronecker’s] cause a stir in the mathematical world. ‘Of what use,’ said Kronecker to Lindemann, ‘is your beautiful investigation regarding π? *Why study such problems, since irrational numbers are non-existent?*’ So back we are once more at a logical scandal such as troubled the Greeks. The Greeks survived and conquered it, and so shall we. At any rate, it is all a sign of the eternal freshness of mathematics.
> —Turnbull, 1951 [1929], I:168, italics added. (The allusion is to Lindemann’s 1882 demonstration of π’s transcendence.)

Most of the issues raised in the present article turn out to be distinctly Kroneckeresque, but that is beside the point here.

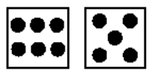
Given all the double-think and shenanigans (such as Hawking’s shell game about God and the Integers as described above), how can we break through to the numeration system itself and clear the air? For that purpose, let’s turn now to a thought experiment about the number eleven. (For this thought experiment, I take my inspiration from Rahman 2014:66.) Think of ‘eleven’, please, then match one of the items in Figure 5 to your mental image of it.

Items [a] through [f] are some of the possible responses to “Think of eleven,” varied by occupation (or preoccupation); psychology; temperament; nationality; and so on. There is no ‘wrong answer’; sadly for us simians, neither is there a right answer, for all of them look equally awkward and irrelevant against a cosmic backdrop.

[a] 11

[b] B

[c] 1011

[d] 

[e] ●●●●● ●●●●● ●

[f] ‘eleven’ (or ‘*unsprezece*’ in Romanian, etc.)

The exercise was inspired by “Think of ‘elephant,’ think of ‘fifteen’ ” in Rahman 2014:66 (which in turn might be an allusion to “I can’t hold 15 in my hand” in Clegg 2003:152?)

Figure 5: What is ‘eleven’?

For most of us, the answer will be [a]. A firmware engineer, if working long hours recently, might choose [b], where we show eleven in hexadecimal, or conceivably [c], where we express eleven in binary notation. Similarly, after a busy night at the casino a gambler might plausibly envision [d]. Someone on the autism spectrum, inclined to think visually about everything, might respond with [e], where my intention is to represent his/her mental picture of, say, marbles, conveniently subgrouped. Finally, to complete the list, a purely literary person (or a smart-aleck) might think ‘eleven’ (or *unsprezece* in Romanian). Fine. There are no wrong answers — yet.

Now we turn to the real question: Of the half-dozen choices, which would the Creator (or, if you prefer, an advanced extraterrestrial) point to as *the* number eleven itself? On reflection, we see that none of our answers fits the bill as *the* number eleven itself. The base-10 notion that undergirds ‘11, 12, 13...’ is the legacy of our ancestors who bequeathed us two thumbs and eight fingers, echoed by ten toes. No gift from *der liebe Gott* there, only branch-grabbing happenstance. Nor can ‘B’ lay claim to being the actual eleven, given the abstruse nature and historically late formulation of its hexadecimal base. Ditto for binary. As for marbles in a queue, intuitively one would argue that such is not abstract *enough* to be a serious contender. (Cf. Jourdain pp. 21 and 35 on ‘symbols of number.’) Likewise ‘eleven.’ Candidly, we haven’t a clue how the genuine number eleven should be represented even here on earth, never mind on a higher plane, or in God's House.

Taking it a step further: In the cosmic eye, is there even such a thing as the counting numbers (1, 2, 3...) by whatever names/symbols/axiomatic scheme? True, we can readily point to *indirect* demonstrations that we are 'thinking correctly,' such as 'making an atomic bomb' or 'navigating to the moon,' but where is the *direct* confirmation that our counting numbers are real, not just an artifact of the culture on monkey-planet? Suppose we cite the composition of the chemical elements, with 1, 2, 3 protons defining hydrogen, helium, lithium...? There I believe we might have our first (and only?) sign of encouragement, reflected back from the universe itself.

Concluding Remarks

Getting real: "But like a diseased financial giant, isn't the Institution of Pi 'too big to fail'?" Point taken. Accordingly, I propose that the scalpel be wielded carefully, as follows: In physics and engineering, the symbol 'π' should stay since it causes no harm there, in its role as a practical tool. The place where its use should be discouraged is, ironically, within the citadel of Queen Mathematics herself. There, to counteract 20-odd centuries of crypto-mysticism and double-think, π should be replaced, in the vast majority of situations, by '3.14' or '3.1415' or 'the 3.1415-algorithm' or 'the 3.1415 algorithm family.' And similarly for ϕ, $\sqrt{2}$ and *e*. After all, 3.1415, 1.6180, 1.4142 and 2.7182 all have strong personalities and are quite capable of 'announcing themselves' on the instant, without the aid of religious iconography.

Similarly, one might argue that '∞' as used in physics (e.g., to define the maximum energy level for an electron, opposite of its ground state energy) is a useful tool. (Note the term 'pure mathematics' in the Nagel/Newman passage quoted in the Prologue.) But elsewhere, there are numerous situations where traditional 'lim n→∞' should be replaced by 'algorithm, for ∃' or 'algorithm, for क ' (borrowing the Devanagari letter *ka* that occurs in *kalp* 'kalpa' and *anant kaal* 'eternity'). That way, the distinction between an eternally living asymptotic curve and its dead abstract limit on the whiteboard is not smeared about for the simian convenience of 'getting an answer' or 'breaking for lunch.'

In sorting this out, it helps also to make a three-way distinction between *tags*, *symbols* and *names*. The term 'mileage' is understood as a useful *tag*, nothing more or less. I.e., no one believes that 'the mileage on a Honda' denotes an actual *thing*. The symbol π should be used the same way, and so it is in physics and engineering, where it plays the role of pragmatic *tag*. But in mathematics, the symbol π has become more like the name 'Santa Claus' where some who hear the *name* mistake it for evidence of the existence of a personage or a *thing*.

Above I have made engineering out to be one of the 'good guys' but in fairness we should note also the following negative role that engineering has played in the story:

> Far from pleasing the faculty councils at the École, Cauchy's *Calcul Infinitésimal* incurred their wrath for being too theoretical and not sufficiently practical. In fact, in late 1823, the Minister of the Interior appointed a commission including Laplace and Poisson to ensure that the instruction in mathematics was attuned to the needs of engineering students. For the rest of the decade the École's administration continually monitored Cauchy's lectures to guarantee their suitability for engineering students. —Hawking 2005:641

This provides an important clue as to why certain concepts in the area of limits and asymptotes were gradually degraded in mathematics, so that Cauchy's 'converge toward' was supplanted by 'converge to', the flea-hop, etc., as described above. It seems likely that continuing pressure from Engineering departments, not just in France in the 1800s but everywhere for the ensuing two centuries, drove those changes.

Note that many parts of conventional mathematics remain untouched by the 'assault' above on some sacred cows. Left standing, as it were, are many cherished parts of the landscape, including the ζ-function (see de Sautoy, *The Music of the Primes*); *i* (see Nahin); the Mandelbrot set; bifurcation nodes as they relate to Feigenbaum's 4.669201 (see reference above to Ekeland 1988, Appendix 2); calculus as the celebration of curves (Boyce 2013, Chapters III-IV); and, as discussed already in connection with Ramanujan, the *right-hand* side of the so-called 'π equation' of your choice.

On the long-term prospects for our 'Queen of the Sciences' (Newman I:294, quoting Gauss in praise of mathematics). Once upon a time, the word *alchemy* simply meant 'chemistry' (from Greek *khēmia*, 'the art of transmuting metals,' by way of Arabic *alkīmiyā*). Do present-day chemists look down upon the alchemists of yore? Not exactly. Theirs is accepted as an important protoscientific tradition. Still, some of their specific pursuits are frowned upon, even ridiculed, e.g., taking quicksilver (Hg) as an elixir (presumably because 'quick' means 'living'). And in forming an overall impression of alchemy today, it is difficult to avoid the words *quaint* and *wrongheaded*, somewhere in the mix. I predict that present-day mathematics, if it does not clean up its act, will suffer a fate similar to that of alchemy. A few centuries hence, it will be abandoned in favor of something else (computer science is a good candidate), and remembered only as an important phase in our simian development, now too quaint to warrant much attention, except by historians.

References

Berlinski, D. (1995). *A Tour of the Calculus*. Vintage Books.

Boyce, C. (2013). *A Calculus Oasis*. Lulu.com

_______. (2018). 'Totemic π, ϕ and √2: Twenty centuries of double-think are enough,' unpublished article. Filename: Pi_phi_root2_double-think.pdf.

Cauchy, A.-L. (1823). *Résumé des Leçons sur le Calcul Infinitésimal*, commentary by Hawking and excerpts translated by J. Anders in: Hawking (2005) 635-667.

Clegg, Brian. (2003). *Infinity*. New York: Carroll and Graf.

Conway, J. and Guy, R. (1996). *The Book of Numbers*. New York: Springer-Verlag.

Dauben, J. (1990 [1979]). *Georg Cantor: His Mathematics and Philosophy of the Infinite*. Princeton University Press.

Dedekind, R., Beman, W. tr. (1901[1872]). Continuity and Irrational Numbers, in: *Essays on the Theory of Numbers*. Dover 1963:1-27.

Dedekind, R., Beman, W. tr. (1901[1888]). The Nature and Meaning of Numbers, in: *Essays on the Theory of Numbers*. Dover 1963:31-115.

de Sautoy, M. (2003). *The Music of the Primes*. Harper-Collins.

Devaney, R. (1990). *Chaos, Fractals, and Dynamics*. Menlo Park: Addison-Wesley.

Ekeland, I. (1988). *Mathematics and the Unexpected*. University of Chicago Press.

Gullberg, J. (1997). *Mathematics from the Birth of Numbers*. New York: W.W. Norton.

Hawking, S., ed., commentaries (2005). *God Created the Integers*. Philadelphia: Running Press.

Hughes-Hallett, D., Gleason, A., *et al.* (2012). *Calculus*. New York: John Wiley, Sixth Edition.

Hungerford, T. (2004). *Contemporary Precalculus.* Belmont: Brooks/Cole-Thomson Learning.

Joseph, G. G. (2011 [1991]). *The Crest of the Peacock: Non-European Roots of Mathematics*. Princeton University Press, Third Edition.

Jourdain, P. E. (2013 [1913]). *The Nature of Mathematics*. New York: Dover.

math.stackexchange.com/questions/14115/Motivation-for-Ramanujans-mysterious-pi-formula [accessed 5 January 2018].

Nagel, E., Newman, J. (1958). *Gödel's Proof*. New York University Press.

Nahin, P. (2010). *An Imaginary Tale: The Story of* $\sqrt{-1}$. Princeton University Press.

Newman, J. ed. (1956). *The World of Mathematics*. Simon and Schuster.

O'Connor, J. and Robertson, E. (1999). *Al-Uqlidisi Biography* available at www-history.mcs.st-andrews.ac.uk/Biographies/Al-Uqlidisi.html accessed 30 Nov 2014.

Penrose, R. (1989). *The Emperor's New Mind*. Oxford University Press.

Poundstone, W. (1985). *The Recursive Universe*: *Cosmic Complexity and the Limits of Scientific Knowledge*. New York: William Morrow.

Protter, M. and Morrey, C. (1991). *A First Course in Real Analysis*. New York: Springer-Verlag, Second Edition.

Rahman, Z. H. (2014). *In the light of what we know*. New York: Farrar, Straus and Giroux.

Rothstein, E. (1995). *Emblems of Mind: The Inner Life of Music and Mathematics*. New York: Random House.

Salas, S. and Hille, E. (1990). *Calculus*: *One and Several Variables*. New York: John Wiley, Sixth Edition.

Serway, R. A. and Jewett, J. W. (2004). *Physics for Scientists and Engineers*. Belmont, CA: Brooks/Cole, Sixth Edition.

Sommerfeld, A. (1939[1929]). *Atombau und Spektrallinien*. Volume II, 2nd Ed. Braunschweig.

Stewart, I. (2008). *Taming the Infinite*. London: Quercus.

Stewart, J. (2012). *Calculus*. Pacific Grove: Brooks/Cole, Seventh Edition.

Turnbull, H. W. (1951). 'The Great Mathematicians' in: Newman I:75-168.

Wigner, E. (1960). The Unreasonable Effectiveness of Mathematics in the Natural Sciences, *Communications in Pure and Applied Mathematics*, 13, 1-14.

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