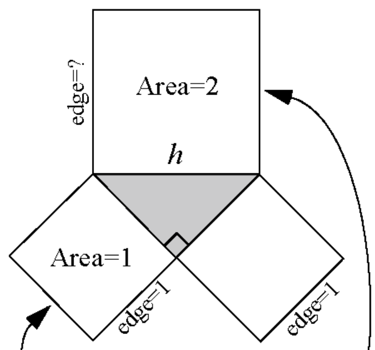
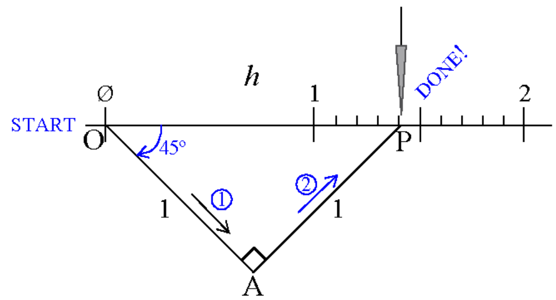


Panel a

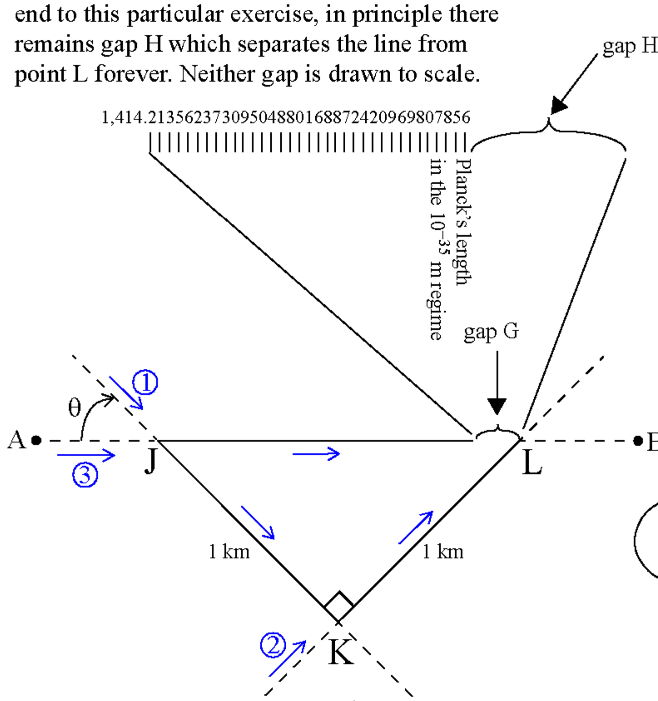
How the topic we know as '√2' first arose: Not as 'the unit square diagonal' per se but in terms of the relation of a *two*-unit square to its corresponding one-unit square. If the edge value for each 'leg' of this diagram is 1, then the area of the 'head' must be 2. But what is *its* edge value, and can that value (*h*) be expressed as the ratio of two integers? No. A crisis ensues for the Pythagoreans.



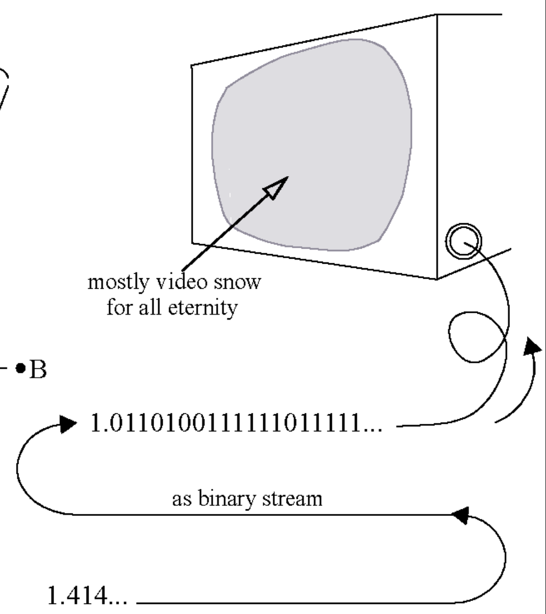
Panel b

This is known as the 'refuge in geometry'; Priestley 58. (Actually, it is an attempt to take refuge in an approach that is even *more* geometrical than that of Panel a.) Starting from O, swing down 45° and draw a right triangle whose first two sides are 1. The second side of the triangle 'finds' P on the number line 'automatically.' The number line itself then provides PO as the third side of the triangle. Thus one has (supposedly) recreated the triangle of Panel a, and may stop worrying about the length of its hypotenuse, *h*. By extrapolation from sides OA and AP, one now knows 'the unit square diagonal' and can even 'see its value' right there at P. Or so the story goes, absent Panel c.

For the narrative pertaining to baseline AB, etc., in Panel c, please refer to the text. The initial gap G is reduced by moving 35 decimals beyond 1,414 m (or 1.414 km). While the Planck length puts an end to this particular exercise, in principle there remains gap H which separates the line from point L forever. Neither gap is drawn to scale.



Panel c



Panel d

Figure 1: Legends and truths of the 1.4142-algorithm(s)

Notes pertaining to Figure 1a: For the flavor of the historical discussion, see Priestley 1998:56-64 or Clegg 2003:34, 61-67. For an excellent account of the original reasoning itself, see Hawking 2005:2-3. (In the historical discussion, a Pythagorean right-triangle would have been firmly in *mind*, but such is not referenced explicitly; in Figure 1a, I employ a Pythagorean right-triangle for expedience, to streamline the presentation.)

Figure 1b depicts the Greeks' well-known 'refuge in geometry' (Priestley 58). With Figure 1c, I show why the 'refuge' of 1b turns out to be an illusion. Figure 1c is the basis for a thought experiment, as follows: Imagine three tractors on a broad flat plateau. Trailing each tractor is a needle-sharp stylus. These custom-built tractors can advance *only* by the increments of the 1.4142-algorithm. On this kind of tractor, the odometer is comprised of a set of 30-odd cylinders (or 'wheels'). A given cylinder has only two positions, one showing its initial zero, the other its assigned digit in the series. Thus, the first four cylinders of the odometer might show [1].[4][1][**0**] or [1].[4][1][**4**] but could not display [1].[4][1][3] or [1].[4][1][7]. By fiat, we claim these tractors to possess God-like precision: if the odometer reads [1].[4][1][4][2][1], then we know with confidence that 1.41421000000 kilometers *precisely* have been traversed, and so on.

After the baseline AB is established, and a 45^{o} angle from it has been measured (indicated by 'θ' in Figure 1c), Tractor One crosses AB at an arbitrary point, call it J. From there, Tractor One traces out the 1-kilometer line that we've labeled JK. After measurement of a 90^{o} angle at K, Tractor Two traces out the 1-kilometer line KL. We regard line AB as a surveyor's baseline only; to finish ΔJKL properly, and to investigate the actual distance between J and L, we now wish to lay off the third side explicitly, starting from J. For this task, we have Tractor Three standing by. Its driver is to make her first stop when the odometer reads 1.414 km then await further instructions. At that juncture, it will appear to bystanders that the triangle has closed on point L . (As for the tractor itself, it has halted in the vicinity of B, let's say.) But as the bystanders move closer, they realize that the stylus did not reach point L. In fact, there is a discrepancy measuring 21.3 cm. Can we fix it? The driver is asked to nudge the tractor forward by its next two hard-wired increments, meaning: from 1.414 to 1.4142 km, and thence to 1.41421 km. Now the discrepancy has shrunk from 21.3 cm to a mere 3 mm. Shall we call that a reasonable Margin of Error given the 1-kilometer scale of the project and all go home? No. Recall that by fiat, each of these machines is deemed to possess God-like precision; there shall be no talk of errors. Then shall we say that we have demonstrated ‘the square root of two to five places of accuracy’? No; that would be a false statement, logically, since 2 has *no* square root. The correct interpretation of the 3mm, one that is admittedly nonintuitive, is the following: Those three millimeters are a window on eternity. We have reached Step Six only, in a series that must go on — not ‘*to* infinity’ but *for* eternity, never attaining point L, indeed *forbidden* (by all members of the 1.4142-algorithm family) to reach point L.

Ergo, there is *no* triangle JKL in Figure 1c, and we should now suspect that there must have been something delusory about the triangle that we labeled OAP in 1b. Yes, OAP is *a* triangle, but it is not *the* triangle that the chorused voices of a twenty-century tradition keep telling us it is. Our ‘automatic discovery’ of point P was easy; *so* easy that it should have raised the red flag of self-deception. Instead, we accepted segment PO of the number line to conceptually ‘complete the triangle’; that turns out to have been a cheat.

“My compass needle points to where Santa Claus lives,” says the child. Similarly, point P in Figure 1b is supposed to show us “where $\sqrt{2}$ is” but that train of thought is reasonable only *if* $\sqrt{2}$ exists. The Square Root of Two, like Santa Claus, has ‘existence’ yes, but only by *magical* thinking, by *childish* yearning. Absent anything to find, there is nothing that point P can possibly ‘show’ us. The following passage in Dedekind is related to Figure 1b, with particular reference to our discredited length PO:

> But the ancient Greeks already knew [...] that there are lengths incommensurable with a given unit of length [e.g., the unit square diagonal]. If we lay off such a length from the point *o* upon the line we obtain an end-point [*p*] which corresponds to no rational number. [And since] there are infinitely many lengths which are [similarly] incommensurable [in just this way, the] straight line *L* is infinitely richer in point-individuals than the domain *R* of rational numbers in number-individuals.
> —Dedekind 1963[1872]:8-9. Concordance: This passage can be found also in Hawking 2005:916 and in Newman 1956 I:528.

But all irrationals are processes; i.e., they are *not* ‘number-individuals’ in Dedekind’s parlance; they are not what we are calling ‘objects’. So his acceptance of the length *op* as the value of an irrational is a fatal flaw in the edifice he is trying to build. Yes *op* is incommensurable but it has a problem that is far more serious than that: the refuge-in-geometry tradition accepts it as the value of something that *does not exist*: the square root of two. It is a tradition that sabotages Dedekind’s whole effort. It means his scrupulous development of a double-*Schnitt* technique

(13) for use in giving continuity to $\mathfrak{R}$ (19-20) was a fool's errand, a 'solution' to a certain brand of (dis)continuity that fails to exist. (Since the square root of two does not exist, it can hardly be blamed as something that interferes with the continuity of *any* number line. To pursue that course would be to engage in more magical thinking or double-think.) And on reflection, we see that the mistake on page 9 of Dedekind has ripple effects back into earlier pages as well, for we see now that there exists a '$\frac{1}{3}$ problem' which significantly weakens Dedekind's cutting technique, even when applied to *R*. This '$\frac{1}{3}$ problem' will be discussed separately in connection with Figures 2 and 3.

"Fine," you say, "Tractor Three starting from J in Figure 1c can never reach L, but isn't that just a rehash of Xeno?" Not quite. In the details of gap G we find a strong *affinity to* the Xeno puzzles, yes, but here we are dealing with something substantive — the 1.4142-algorithms which have been run on computers and studied by thousands of researchers worldwide, whereas Zeno was just *inventing* scenarios to stimulate discussion. And there is another difference: Thanks to Planck, we now know of scenarios in which Xeno's halving limbo would be mercifully curtailed. Each of the 35 hash marks within gap G represents a slight additional advance by Tractor Three, beyond its initial halting point of 1.414 km. In *principle*, this kind of 'progress' goes on forever; but because the Planck length is an absolute practical (physical) limit, we are 'saved from that fate' in this particular scenario. Having 'run out of granularity,' Tractor Three must stop after a mere 35 decimals (counting from 1,414 meters in lieu of 1.414 kilometers, that is). But we are far from done.

In Figure 1d, we learn that there is something even worse than Xeno's quirky limbo: a kind of 'video-snow hell' from which there is no escape. I call it 'hell' because interspersed with eons of video-snow, images of 'everything' would *also* be generated eventually, all of them 'fake' of course, yet close *enough* to the face of [name your person of renown or infamy] to catch one's attention and break the monotony of, say, the preceding seven thousand years of steady video snow. Who would (or even *could*) be subjected to such a terrible punishment? A cyborg (or an 'AI' in today's vernacular) convicted of a felony, that's who. (In Poundstone 1985:230-231 there is a passage about pi as a random-number generator that was the basis for my Figure 1d scenario.)

Once we get beyond the $\sqrt{2}$ specifics of Figures 1a and 1b, the two-part message of Figures 1c and 1d can be generalized to apply to π and ϕ as well. Elsewhere (Boyce 2018) I have drawn the corresponding graphics for those cases, each of which includes its own attempt at a 'refuge in geometry'; each of which proceeds to the discovery of a 'gap G' that reveals the 'refuge' to have been delusional. For the corresponding π and ϕ thought-experiments, again there would be a God-perfect tractor with a specialized odometer and gear box that knows only one of the 3.1415-algorithms or only one of the 1.6180-algorithms. In both cases, there would be a half-km rod made of MgLi alloy with a stylus at one end and pivot at the other. The tractor would pull the rod around on its pivot to draw an arc. In the one case, the arc would reach a length of 3.1415926 km or slightly longer, but always with a gap that would prevent an actual circle being drawn (assuming a God-warrantied diameter of precisely 1 km). In the other case (for ϕ), using a variation on the construction of Eudoxus, the tractor's arc would 'discover point P' on a baseline AZ, the one that is supposed to be the Golden Cut. Trouble would ensue when we tried to measure segment PA by driving the tractor east to west, back toward the origin at A. There would always be a gap between the stylus and point A, and this would tell us that point P could not have been the Golden Cut after all. (Indeed, from that exercise we would come to understand that there is *no such thing* as the Golden Cut, just as there is no such thing as 'the square root of 2.') In both the 3.1415 scenario and the 1.6180 scenario, one could again escape from Xeno's limbo by invoking Planck's length at the 35th decimal (after converting *km* to *m*). But again, there would be a corresponding video hell offering *no* such escape: output from the 3.1415 algorithm of choice or from the 1.6180 algorithm of choice, each of which is likewise a random number generator, could be fed as binary into a TV monitor that runs for eternity, creating all possible images eventually.

Time out, to acknowledge some examples of *residual* beauty still to be found in the π cottage industry and in the ϕ cottage industry, our exposé of their religious and double-thinking aspects notwithstanding. In the case of π, the trick would be to stay focused on the *right-hand* side of a so-called π equation; in the case of ϕ (or ϕ′), the trick would be to focus on an actual *number*, such as 0.618033988, as distinct from the hocus-pocus of 'the constant ϕ' or the Golden [Whatever].

The right-hand side is beautiful, but the left-hand side is problematic.

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}}$$

$$\frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}} \text{ yields } \frac{1}{3.14...}$$

The left-hand side needs to be scrapped and replaced by an honest statement placed over here.

Here, from math.stackexchange.com (see References), I have copied an especially beautiful π equation, one that many readers will recognize immediately as the work of Ramanujan. Now the right-hand side of the original is beyond reproach (and perhaps beyond understanding; certainly I do not claim to fathom it), but the 'π' and '=' turn the overall statement into a lie, in just that way that *any* kind of 'π equation' must be a lie, even this especially attractive one. To fix it, I've rewritten it (in the second line) with the left and right sides swapped, and with π replaced by '3.14....' Note also the absence of an equals sign in the emended version. (For more such examples, see the row in Figure 4 where I introduce the symbol ¥ for *yields*.)

Second example of residual beauty: Even though 'Golden Cut' taken *as* the name of a number is a fraud — for failing to acknowledge the primacy of the 1.680-algortihm, the sole reality — it is still possible to do legitimate things with some *actual* numbers that are ϕ-related. The difference might seem subtle, but it is crucial. Inside the world of the bifurcation diagram, the *number* 0.618033988 (aka the first nine decimal digits of phi-prime, ϕ') holds a special place; see Ekeland, Appendix 2, pp. 135-136, where he employs that number, $\pm$ 0.000000001, to demonstrate the Lorenz 'butterfly effect' (pp. 64-66). Similarly, δ, the so-called Feigenbaum 'constant' is no such thing, but the *number* 4.669201 is indeed tied to (derived from) certain bifurcation points inside the logistic map that are real. No arguing with that *aspect* of the Feigenbaum story. (I am grateful to Philip J. Stewart for reminding me indirectly [private communication, 2018] of the Ekeland book, which had been one of my treasures — at the Minneapolis Public Library — some thirty years ago.)

As for ‘the constant *e*’, there we find no counterparts to Figures 1a, 1b or 1c, but Figure 1d does apply since any 2.7182-algorithm is, once again, a random-number generator. At a certain level of abstraction, all four algorithms are delivering a single message to us earthbound simians: In the *universe*, time is more important than space. Durations are key, distances are trivial. (Note the parallel here with eternity vs. infinity to be taken up in Part Two.) Once understood, the horrors of Figure 1d should cure anyone of their π, ϕ, $\sqrt{2}$ or e infatuation.

In the title and elsewhere, I invoke Orwell’s term *double-think*. I find it apt for the following reason: With one part of the mind, the mathematician seems to be fooled by the slick-looking ‘authority’ of the $\sqrt{2}$ icon itself into believing there really is a square root of two ‘out there somewhere.’ But simultaneously, in some deeper recess of the mind, surely there is grudging acknowledgement that two has *no* root. Consider the following passage: “[T]he sequence 1, 1 + 4/10, 1 + 4/10 + 1/100 [...] or 1.4142..., got by extracting the square root of 2 by the known process of decimal arithmetic, has [no limit.] [If it did,] and it were denoted by ‘*x*’, we would have x^2 = 2.” This is from page 59 in Jourdain’s classic. First he extracts “the square root of 2 by [a] known process”; then, in the same breath, he allows that there is no such thing as the square root of 2. (An obliquely related item: see Protter/Morrey 13 re the existence / nonexistence of $\sqrt{2}$.) An especially good example of mathematical double-think is found in de Sautoy 67 (original emphasis):

> To capture the impossibility of expressing such numbers in any way other than as solutions to equations such as $x^2 = 2$, mathematicians called them *irrational numbers* [...] Nevertheless, there was still a sense of the reality of these numbers since they could be *seen* as points [on] the number line. The square root of 2, for example, is a point somewhere between 1.4 and 1.5. If one could make a perfect Pythagorean right-angled triangle with the two short sides one unit long, then the location of this irrational number could be determined by laying the long side against the ruler and marking off the length.

The above passage, written by a professor of mathematics at a world-class university, is rubbish through and through. Still, one values it in its role of ‘smoking gun’; it helps make explicit what would otherwise have to be inferred from a thousand tiny hints in the language of mathematics elsewhere, such as the *approximate answer* / *exact answer* argot of the math classroom. As prelude to that important subtopic, note how the word *exact* is used (abused) in the following three passages:

"Kepler pointed out that the ratios of consecutive Fibonacci numbers approach 1.618... The *exact limit* is the golden number, τ" and "τ = 1.61803398... Its *exact value* is $(1 - \sqrt{5})/2$"; in Conway/Guy 1996:112, 184, italics added. "[T]he circle being considered as a polygon of a great number of sides, its area ought to equal the product of the circumference into half the radius. Now, this result is *exactly true*"; Jourdain 2013[1913]:37, italics added. Thus, knowing in his/her heart of hearts that π does not exist, the math teacher feels justified by long tradition in speaking nevertheless of 'the constant π' and 'the true value of π.' And perhaps s/he even harbors a Secret Wish along these lines: "I pray that the decimals of π stop around the octillionth place, for only then would we teachers be vindicated in praising π as an 'exact answer' on exam papers while marking '3.1415926' wrong for being 'only approximate'." Thus, it appears that double-think is not just an ad hoc refuge for the mathematician; it is an ingrained habit, lodged deep in the very 'DNA' of the field.

Another example of dogma by double-think is the assertion that C:D=π. In one part of the brain, the math instructor knows very well that with D set to 1, C is *in*commensurable and *ir*rational. Therefore, C has no business being shown in any *ratio*, especially not this (supposedly) foundational ratio upon which so many spired castles have been erected. But in some other corner of the brain, one cleaves to the ancient rule-of-thumb that says "Pi is *pretty darned* ***much*** like the ratio of a circle to its diameter, isn't it?" Thus, a piece of sanctioned nonsense that finds its way into every textbook. From the existing level of voodoo, it would be but a short step to the following: "The constant π is a gold ornament displayed on black velvet. God waits patiently for us to find it in its sealed vault at the End of the Universe."

While we are still adrift in outer space, let's take this opportunity to consider the argument of A Perfect Circle by Divine Fiat: "Suppose that God employs her Big Compass to draw a circle in the cosmos, and declares it to possess a circumference equal to π light years *exactly*, by royal fiat. In this way, has She not defeated your argument?" No. She has merely shifted the trouble from C to D. Inside that perfect circle of circumference C you will find no diameter, D, only a line whose approximate length is 0.99 light years, and which seems to 'grow' each time you try to measure it 'more accurately,' even as it fails eternally to get 'where it is going.' Therefore, I repeat, any definition that relies on 'C:D' is a lie.

In wrestling with this obdurate gap, which first makes a mockery of our quest for C, and now plays havoc with God's own diameter, D, one senses a distant relative of the Blurriness Relation (*Unschärfe-Relation*), which was later named the *Unbestimmtheit-Prinzip* (Indeterminacy Principle; Sommerfeld II:196-201), only to be mistranslated by journalists as the Uncertainty Principle which, despite its inanity holds sway by dint of its ubiquity.

Sidebar on Hui Shi (370-310 BCE): "If a foot-long stick is halved daily, not in myriads of eons will it ever be exhausted." This sentence is found in Chapter 33 of the *Zhuangzi* [my translation], where it is attributed to Hui Shi. (Others attribute it to Gongsun Long; see baidu.com.) On that same page in the *Zhuangzi* there are a dozen pithy quotes, ranging from serious, Zeno-like paradox (which seems to be the intent of the 'stick' passage) to leg-pulling paradox, to Ionesco-flavored mind-cleansing nonsense ('Chickens have three legs'). Among them we find: 'A compass cannot make circles' (规不可以为圆). In the context of this article, with its exposé of the fairy-tale of '$C:D=\pi$', the latter saying has special appeal: In a giddy mood, a wisecracking philosopher of the Warring States period said something that turns out to be a genuine, long-term philosophical problem.

Earlier in Figure 1c, in our attempt to draw ΔJKL, we divorced ourselves from the number line that plays such an important role in Figure 1b. Now to finish the story we must return temporarily to the number line because it was concern over its 'continuity' that motivated Dedekind to formulate his (double-)*Schnitt* concept — for the express purpose of accommodating $\sqrt{2}$, π *et al*. Please refer to Figure 2. Case A depicts 1.4142... as an object that has been accommodated on the number line after Dedekind's 'cut' mechanism has been extended from R to $\mathfrak{R}$ (Dedekind 8-9, 19-20). Case B depicts 1.4142... as a process that cannot, by definition, have anything to do with the number line, so we give it its own (arbitrarily oriented) scale.

We have not yet looked (directly) at the term 'irrational number.' Some authors trace it back to the Greeks themselves, citing the word *alogos* ἄλογος 'unreasonable' in connection with the diagonal of a square (e.g., Turnbull 87). Meanwhile, others might say, "We do not mean 'irrational' in the sense of 'unreasonable' or 'crazy', only that the number in question is non-rational." But the real problem with the term 'irrational number' is with the noun *number*, for nothing so labeled *is* a number. It does not matter what adjective has been placed in *front* of the noun; what matters is the quiet deception of the noun *itself*, with possibly far-reaching consequences, such as the fool's errand that Dedekind pursued so earnestly.

Counterpoint. Rather than accuse the field of double-think, which is my inclination, Rothstein sees only the light of reason in mathematics:

> An instantaneous event, the sum of an infinite series, the precisely calculated irrational number — these gradually came to be treated not as concrete objects [...] but as the results of *processes*, goals for a sort of mathematical yearning. —Rothstein 1995:58, his italics.

And in Conway/Guy we find this: "The square root of two, like other infinite precision real numbers such as e and π, is not really real in the physical sense! They are all figments of the mathematician's mind" (1996:213). Note how these passages stand in a complementary relation to the Nagel/Newman passage quoted in our Prologue: "We repeat [...] *necessary logical consequences*." But such rare moments of candor in the literature (of semi-popular math books and undergraduate textbooks) scarcely tip the balance against the thousand little things that whisper 'double-think' as in Cases A and B of Figure 2; as in the math class argot of 'exact' (discussed earlier in connection with Conway/Guy 112 and 184) and 'approximate.' See also the word 'exact' as used in Jourdain, *The Nature of Mathematics* (2013[1913]): 'exactly true' (11, 37, 39); '*exactly right*' (50); 'perfectly exact' (53). Whence his preoccupation with *quantities* alleged to be 'exact' if the name of the game truly is logic, questing after Logic? Cf. Jourdain, Chapter VII, 64-68.

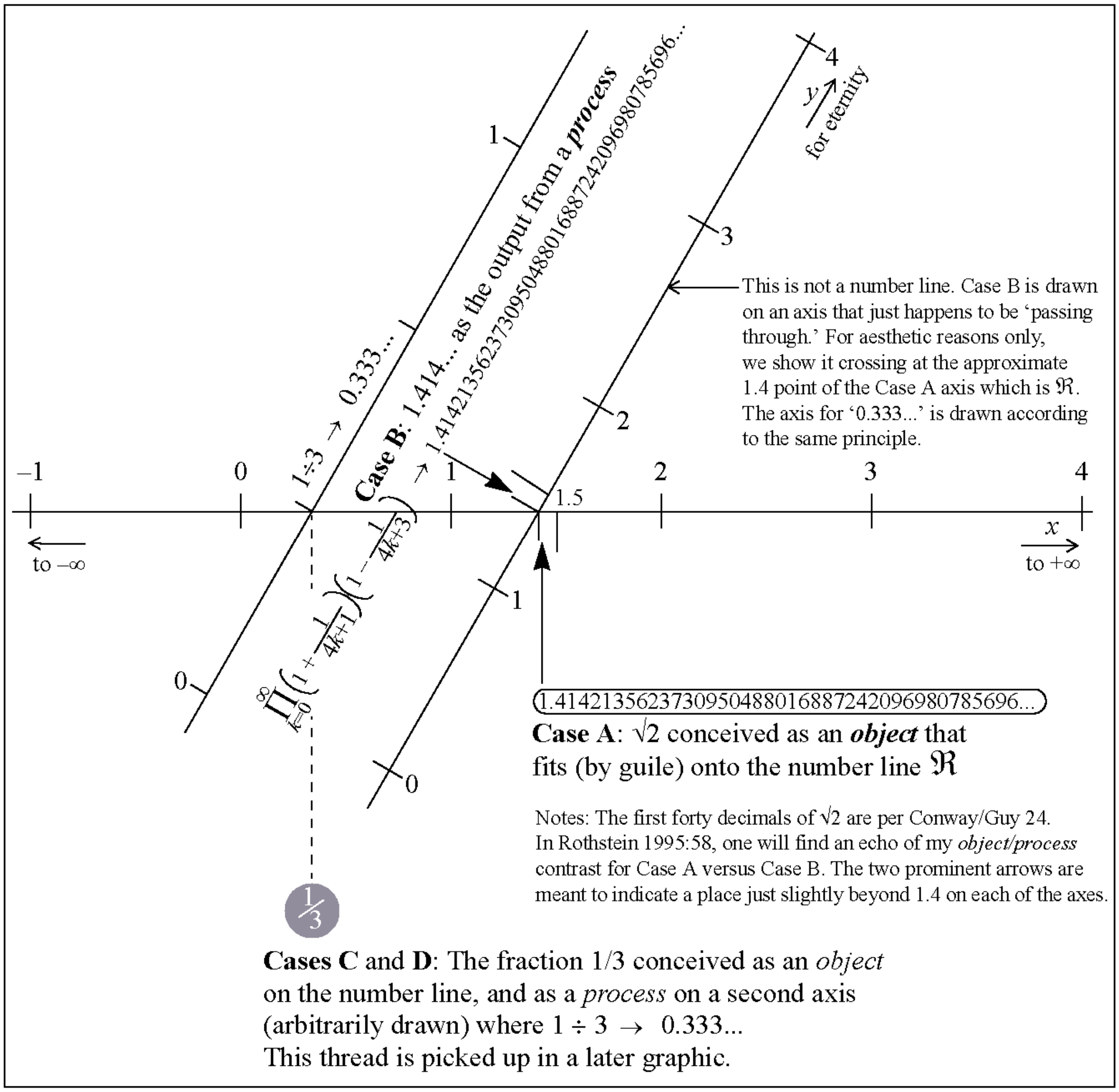
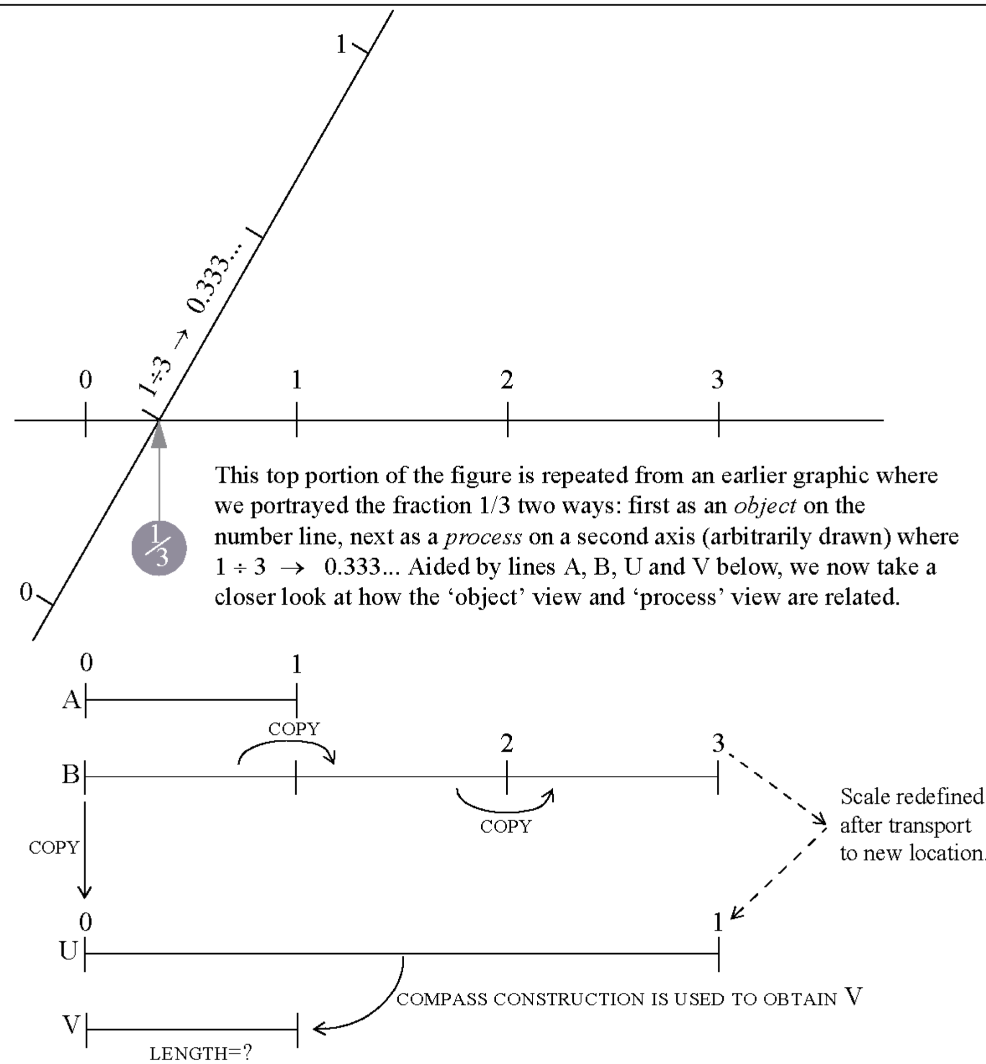


Figure 2: Case A (1.4142... as an object) and Case B (1.4142... as a process)

With Cases C and D in Figure 2, we introduce the question of whether 'one-third' is a static object (as when $\frac{1}{3}$ denotes one whole marble in a set of three whole marbles) or a dynamic process (as when we contemplate $1 \div 3 = 0.333...$ in perpetuity). Through Figure 3 we learn that these two Janus faces of the 'one-third' concept cannot be reconciled.



Thought experiment: Rod **A** is a given. We define its length as 1 unit. Then we use it to construct rod **B** as indicated by the two curved arrows. Since rod B has length 3, the A:B relation must be *precisely* 1:3, i.e., ***a perfect* 1/3**. At the request of a colleague, we now make a copy of rod B and call it **U**. The colleague moves rod U to a new location far away, beyond the reach of any communication with us. Later, she has occasion to make a rod that will be 1/3 the length of U. She starts by assigning U a new scale: its length is declared to be 1. She uses a Hellenic compass construction (not shown) to trisect a line U and thus produce line **V** whose length is one third the length of U. From this she creates rod V as required. But what is the *measure* of rod V? She needs to record that too. However, to find *that*, she has no choice but to resort to the following arithmetic problem: 1 ÷ 3 = ? Answer: 0.333... (forever).
Now, from the diagram above, you and I can see that the U:V relation replicates the B:A relation, right down to the pixel. For just as A is precisely 1/3 of B, so V is precisely 1/3 of U. And yet, there is something about our numeration system such that the measure of V, in the context of the thought experiment, must remain an indeterminate, ragged amount that *keeps growing eternally*. Even though 1/3 is a 'rational' number, its true nature appears now to be that of a *process*, in just the way that an '*ir*rational' number is a process, not an object (as argued in the text). Is this, then, the world of mathematics as advertised, or have we stumbled upon yet another of its secrets best left in the closet?

Figure 3: The trouble with thirds

In the conventional narrative, a fraction such as $\frac{1}{3}$ seems substantive when juxtaposed with a surd such as $\sqrt{2}$; after all, $\frac{1}{3}$ is 'rational' — a 1:3 ratio — while $\sqrt{2}$ is *ir*rational. As such, $\frac{1}{3}$ is treated as an object, not as a process (using my nomenclature), and this tacit identification as an object makes it immediately welcome on the number line. However, elsewhere in the literature, we see that $\frac{1}{3}$ may be cast as something in-between an object and a process, or perhaps both at once, as in the following:

> However, *infinite* decimal expressions, such as the *full* non-terminating expansion π = 3.14159265358979... present certain difficulties [to a Turing machine.] [...] It might be felt that it is impossible to contemplate an *entire* infinite expansion, but this is not so. A simple example where one clearly can contemplate the entire sequence is 1/3 = 0.333333333333333... —Penrose 1989:50, 81, his italics.

Here, 0.333... can be made to seem *almost* like an object, *almost* like something that may be 'contemplated' (because 0.33333... looks comparatively simple beside 3.14159...) But an unintended side-effect of the Penrose passage is our realization that '$\frac{1}{3}$' has two 'faces' only one of which belongs on a number line. In other words, not only is Dedekind's $\mathfrak{R}$ in trouble because of the practice in mathematics of treating processes as if they were objects (e.g., confounding a 3.1415-algorithm with the static π icon), but *R* too is in trouble since $\frac{1}{3}$, $\frac{2}{7}$, etc. are not quite what they seem at first sight: Each has both an 'object face' (e.g., $\frac{1}{3}$) *and* a 'process face' (0.333...). When $\frac{2}{7}$ means 2 marbles from a set of 7 marbles, it is a static object; when $\frac{2}{7}$ means $2 \div 7$ = 0.2857142857142857..., it is only the name of a process, never mind that it is a perfectly repetitive and predictable process — still it goes forever. That's the point.

After the dust settles, what survives intact of Dedekind's original 'cut' philosophy is a tool for handling (conceptually) such clumps of numbers as the following (each of which I declare, by fiat, to be an object, *not* the output of a process): *S* = {7.98, 7.9841, 7.9841772889, 7.9841772889341912, 7. 98417728893419125...}. I refuse to say there is an infinite 'number' of such objects; however, one may keep adding arbitrary new members to set *S for* eternity, yes. And here we have a genuine problem: How can the burgeoning, vaguely defined members of *S all* fit somehow on the number line, and how one can show that in spite of the notionally unlimited 'disruptions' that *S* may engender, the number line, *R*, itself still has *continuity*? But this meek, limited application of the double-*Schnitt* concept is a far cry from the 'news-making' cut on $\mathfrak{R}$ as originally advertised.

Part Two: The overloading of ‘=’ and abuse of the ‘convergence’ concept

The term ‘operator overloading’ refers to a feature whereby a programmer may enrich the definition of a selected operator. For example, she might redefine ‘+’, the addition operator, to handle both numbers and strings, dependent on the type of the operand pair. Subsequently, both “5 + 7” *and* “ ‘a’+‘b’ ” would be accepted by her C-language compiler as legal expressions. Some disapprove of overloading, so much so that certain languages do not even permit it. But at least it is a well-defined practice about which one may argue pros and cons in the software engineering context. By contrast, the symbol ‘=’ in mathematics seems to have undergone a kind of vague de facto overloading, only by semi-conscious accretion over the centuries. Most conspicuous is the use of ‘=’ for both assignment and equality testing, a comingling of fundamental concepts that would be a recipe for disaster in software engineering. But it does not stop with ‘=’; please refer to Figure 4.

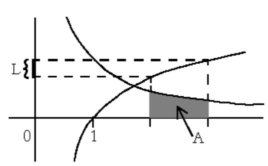
Example	Functionality	Comments
[Let] $n = 1, 2...$	*assignment* of a value or values to n	Given the stark contrast between *assignment* and *equality*, all programming languages make a clear distinction of '=' vs. '= =' (or ':=' vs. '='), respectively. Meanwhile, how does mathematics get by without such symbology? The one symbol '=' is called upon to handle both in a manner that may be termed 'overloading' with the caveat that this particular flavor of overloading developed in a vague, ad hoc manner over the centuries, not as a deliberate strategy à la *operator overloading* in software engineering. Lacking the operator '= =' and not quite trusting overloaded '=' for the job, the mathematics author often feels constrained to test for equality this way: '$a - b = 0$' (Zill 4-5), thus bloating the syntax with gratuitous operands and operators. See text.
[If] $a = b$ [then...]	*equality test* of a and b (i.e., the 'a = = b' test of programming languages)	
$\lim\limits_{x \to a} f(x) = 1$	*copula* for a limit value	In column 1, the '=' corresponds to 'is' (the copula) in the following sentence: "The limit value of function $f(x)$, as x approaches a, is 1" (after Gullberg 356). To say that $f(x)$ *equals* something (= 1) would be to trash the very notion of an asymptotic process, but even if we interpret '=' to mean 'is' (via additional overloading) a problem remains: The term *limit value* is itself bogus, a sleight-of-hand trick: A function may have a value; a limit is simply a limit; and never the twain should meet; but here they do. Why? See discussion in text.
$\lim\limits_{n \to \infty} S_n = 42$ copula in a *limit-surrogate* relation such as: "[The $\zeta(s)$ series] sums to the limit $\frac{\pi^2}{6}$" (Hawking 2005:822) or: "$\zeta(2)$ [...] $= \frac{\pi^2}{6}$" (Conway/Guy 262) $\sum\limits_{n=1}^{\infty} a_n = \lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} a_i$		The first example is after 'sum to infinity' in Gullberg 270. (See also Devaney 12, 109-112.) Protter/Morrey 48 has this: "[$n \to \infty$ is] used to define a limit at infinity." N.B. the phrases '*to* infinity' and '*at* infinity'; contrast '*for* eternity' as discussed in the text. Comment on 'sums *to* the limit' in Hawking 822: A series has *no* such 'sum'; nor can π be part of *any* 'limit' since π is a process, not a thing. One should say that the series converges *toward* $\pi^2/6$, as it (too) runs *for* eternity. (In such contexts, Cauchy used 'converge toward' [*converga vers*], but over time, his precise language has been supplanted by crude rephrasings or/and inept translations. See text.) The example with Σ is from J. Stewart 729; see comments in text.
$e^{i\pi} = -1$	fudging of a supposed *identity* relation	Here we see the famous 'identity' by which e, i and π seem to have been distilled to celestial heights and there set 'equal to -1'. But it turns out that the -1 in that relation is only a limit. See Conway/Guy 254-256 and remarks in text.
$\pi = 4 \times \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \cdots\right]$ $4 \times \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \cdots\right]$ ¥ 3.1415... And to avoid the lie of '$\sqrt{2}$ = ...' write this: $\prod\limits_{k=0}^{\infty} \left(1 + \frac{1}{4k+1}\right)\left(1 - \frac{1}{4k+3}\right)$ ¥ 1.414213... Similarly, to avoid the lie of 'ϕ = ...' one should write this: $\frac{(1+\sqrt{5})}{2}$ ¥ 1.6180... And rather than set something 'equal' to e, one should write: $\lim\limits_{x \to 0+} (1+x)^{1/x}$ ¥ 2.7182...		The first equation says, in effect, "The mathematical 'constant π' will find, at the end of the rainbow, its 'true value,' by means of an infinite series that we pretend is kind-of-asymptotic-like, although we know better, as we thus trivialize the sacred concept of limits." To avoid such abuse of the equals sign and the concept of limits, we need a brand-new symbol, say ¥, meaning '... is a process that yields...', as demonstrated at the left with '...¥ 3.1415...', etc. where four accustomed lies have been turned into honest statements. The fourth example alludes to Protter/ Morrey 121; sources for the others are in the text. Counterexample: In sources such as Gullberg 778 and Hawking 822 and 876, one is encouraged to see references to 'the function $\zeta(s)$' of Euler and Riemann, where the library of 'constants' comprised of nonexistent numbers might otherwise have been 'enriched' by 'the ζ number 1.644' ($\approx \pi^2/6$).
 $\int_a^b f(x)\,dx = \mathtt{F}(b) - \mathtt{F}(a)$ Smack in the middle of the Fundamental Theorem of Calculus, we find what? An *equals* sign. But how can an area A equal a distance L?		Once again, we find ourselves emerged in double-think: On the one hand, Dimensional Analysis is a foundation stone of the sciences; on the other hand, the very thing that makes calculus valuable is lodged in its treatment of dimensions n and n+1 as interchangeable. Thus, one pretends the units of $\mathtt{F}(b) - \mathtt{F}(a)$ are m^2, when really they are m. Or, conversely, by sleight-of-hand one implies that the units of an integration are m when really they are m/s, as in Stewart 2012:335. Yes, one has finessed the situation, but at the cost of trivializing calculus, as though it were just a dubious student crib, not a major discovery. (The illustration of 'A = L' is after Boyce 2013:44-46.)

Figure 4: Some of the half-dozen ways that '=' is overloaded in mathematics

In software engineering, where assignment and equality are like oil and water, the use of a single symbol to handle both would lead to rampant bugginess. The tacit blurring of those two concepts in mathematics manifests in other ways: sometimes in vagueness, sometimes in expressions that are precise but strangely formulated. Take the syntax '$a - b = 0$'. In mathematics, this is a species of equality statement or equality test (e.g., in Zill 4-5). In such an equation, it seems that we are concerned with three entities (a, b and 0) as they relate to two operators, but logically, given the purport of the statement, it should contain only *two* entities and a *single* operator (like 'a = = b' in C or Java for example). Thus, one of the more annoying tics of the language of mathematics.

In row 3, we have "The limit value of function $f(x)$, as x approaches a, is 1" (after Gullberg 356). In precalculus, the concept of a limit is developed very slowly (e.g., over a 50-page span in Hungerford 2004:828-877), with care that verges on the quasi-religious. And with that tradition I have no quarrel since the concept *is* subtle and *is* inherently interesting, and practical to boot. But in the calculus curriculum proper, the concept of a limit is often abused. The trouble comes with asymptotic functions that converge *toward* a limit. While Nature is slow and patient, we simians are the opposite, so a 'culture clash' ensues. A function may have a *value*, and a limit is simply a *limit*; why would one jam the two words together as 'limit value'? That argot of the mathematics classroom is a way of trying to disguise or dignify the practice of abandoning an asymptotic process at an arbitrary time (e.g., lunchtime) to perform a flea-hop *from* the function *onto* its limit. The math instructor may keep the distinction clear in his/her head, but the student will surely just take things at face value, too busy to notice the shell game until months later, if ever.

In the next row of Figure 4 we have 'sum to infinity' (Gullberg 270) and 'sums to the limit $\pi^2/6$' (Hawking 822). Here I append two related passages, not cited in Figure 4 for lack of space: "[It] was a sheer assumption that such a *process* as 1.4142... [...] has any limit at all" (Jourdain 2013 [1913]:59-61, italics added). "[By the epsilon method] we no longer need to take [1 + ½ + ⅓ +...] all the way *to* infinity to get the *value of the sum* to be 2" (Clegg 2003:126, italics added). We may count the Jourdain passage as a brief moment of sanity flashing by in 1913. Conversely, the Clegg passage is like an x-ray in which all parts of the prevalent double-think anatomy are suddenly lit up together. (In that way, it is akin to the 'smoking gun' passage quoted earlier from de Sautoy 67.)

A final note about sums and limits. What could be more straightforward than the summation symbol, Σ? With subscripted i=m and superscripted n, it indicates "the sum of all [terms] as i goes from m to n" (Gullberg 105). Replace the n by ∞, and we have *the* ultimate math icon, used to good effect on the spines of Newman's anthology, for instance, where the volume numbers are indicated by subscripted i=1,2,3,4. However, for the variant that has ∞ on top (see example in row 4 of Figure 4, after Stewart 729), some authors feel compelled to issue a warning: "It is important to note that the sum of a series is not a sum in the ordinary sense. It is a limit." (Salas/Hille 614.) What the warning means is that we are now back in the realm of quicksand and double-think where an eternal series *converges* '*to*' its limit so that the limit can morph into its 'sum'. In such contexts, Cauchy used 'will converge *toward* [*converga vers*] a limit', but over time, his precise language has been supplanted by crude rephrasings, whereby 'toward' morphs into 'to'. See Cauchy as presented directly (albeit in translation) in Hawking 655, 658; then compare Hawking's paraphrase of Cauchy using a 'modern' idiom, on p. 640.

In the so-*called* identity, $e^{i\pi} = -1$, of Euler we find a Holy Trinity of sorts. But of the three, only i is legitimate, which is ironic since i stands for imaginary. (I call i legitimate for reasons explained in Nahin passim.) Meanwhile, e does not exist and π does not exist (as an object or 'constant'). Moreover, even if those two existed as bona fide numbers, the right-hand side, '= –1', turns out *not* to be an identity relation after all (though widely advertised as such). Rather, –1 is a limit; see Conway/Guy 1996:255. Thus, the symbol '=' has once again been overloaded, this time in the role of 'helping to fudge an identity relation' by lying about the right side of an equation (whose left side is also bogus for the reasons mentioned above, but that is not the main point here). The row in Figure 4 devoted to π (and ¥ = *yields*) contains examples that hark back to the Ramanujan equation shown earlier.

Part Three: Our numeration blind spot

We live in (collective) ignorance of what numbers are. While the lay person would object with indignation to that statement, it would come as no surprise to the mathematician. This harks back to the italicized phrase '*purely logical consequences*' in the passage we quoted from Nagel/Newman in the Prologue. But even if Queen Mathematics has summarily swept 'What numbers are' off the table, there is no law that forbids others from expressing curiosity about the crumbs of that question that remain on the floor. But before confronting the topic directly (with help from Figure 5), let's get a feel for how it seems *almost* to exist, *somewhere* in the vastness of the conventional math universe.

At the very moment of its birth (say 1889), *number* theory simultaneously gives a perfunctory nod to the *numeration* question, and promptly washes its hands of it. (Cf. Joseph p. 35. See also the fleeting mention of 'symbols of number' in Jourdain p. 21.) Consider the following reflection on how/why Peano axiomized numbers, taking the 'natural numbers' (aka 'counting numbers') as his foundation:

> It might seem strange that Peano should need to [develop our 'counting numbers' from a set of propositions] but those familiar numbers we use all the time [...] have to come from somewhere [...] it's easy to think of them as *real things* [...] But in reality these numbers [...] are *just symbols* we use to represent the cardinality of a set. I can't hold 15 in my hand. [Peano in 1889] lets us build those numbers in terms of a series of sets that are almost hauled up by their own bootstraps. —Clegg 2003:152, italics added.

Early on, Peano would have been well aware of (indeed driven by) the fact that our numbers are not 'real things,' but once he entered the realm of axioms, all attention was focused there and the philosophical question would have been spirited away to the far horizon, where it became 'not my job' for the mathematician. Let's look at some representative examples to see what *is* typically covered, just 'inches away' from what would be numeration theory, if such existed.

Ian Stewart (2008) devotes pp. 8-19 and 40-53 to an overview of various ancient and exotic number systems. On p. 138, he introduces us to the hierarchal system comprised of the *natural* numbers, the *integers*, *rationals*, *irrationals* and *real* numbers. (With slight variations in nomenclature — e.g., with *cardinal*, *counting* or *concrete* employed as a synonym for *natural* — we find the same model reflected in Clegg pp. 150-152; Gullberg pp. 5, 70-1, 157; Jourdain p. 21; etc.) Similarly, Gullberg 1997 has a chapter entitled 'Systems of Numeration' but its 38 pages are dominated by the ancient and exotic, with our own system of Indo-Arabic numerals covered only in a nuts-and-bolts fashion as he recounts their introduction to the West by Fibonacci in 1202 (page 50). By 'nuts-and-bolts' I mean there is no hint in Gullberg of the kind of philosophical issue raised by Clegg above. A similar approach is found here: Joseph 2011: 30-75, 198-206, 338-339, 460-466. In each case, having arrived by a richly detailed historical path at the (terribly named) 'real number line' upon which so many impressive edifices and sky castles may be built, it feels as if the the whole waterfront surely was covered.

But what about the moment-before-Peano on which Clegg focuses for a moment? It is nowhere to be found in the standard presentations. At first glance one might think that Dedekind's essay on 'The Nature and Meaning of Numbers' (1963[1888] 31-115) might cover the territory that I say is neglected, but his overarching agenda turns out to be quite the opposite: "With reference to this freeing the elements from every other content (abstraction) we are [now] justified in calling numbers a free creation of the human mind" (68; cf. 31). I.e., the essay is all about securing the cloud castle and guaranteeing that it does *not* touch the ground, so to speak. An analogy: After axiomization, the mathematician focuses on the new turrets and spires that extend the citadel upward, never looking back at the foundational integers, just as a construction worker would be focused on the beam where he walks, 57 stories above the sidewalk, not second-guessing details of the building's foundation. But there is no law against someone *else* revisiting the foundational level, either to praise it or question it. That's what we are doing here, as 'outsiders' to the profession.

As we try to pursue this question of how real the numbers are, it doesn't help that Hawking chose *God Created the Integers* as the title of his thousand-page mathematics anthology. The title comes from the following *bon mot* of Kronecker's: "The dear Lord made the integers; all else is the work of humans" (my translation of *Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk* on de.wikipedia). Inside the anthology itself, the title's attribution comes very late (Hawking 2005:892) and with such a perfunctory glance at the historical context that the general reader is still left in the dark as to its purport. Only from other sources, such as Clegg 68, 190-194 or Dauben 66-70, can one learn what Kronecker actually meant: "The mathematics of integers is sufficiently rich to keep us fully occupied. Other types are superfluous to us mortals or simply nonexistent; their pursuit, as by Lindemann and by Cantor, is unseemly and foolish" (my paraphrase).

Clearly, Kronecker was a reactionary, a figure that many would dismiss as having been on the wrong side of history, not part of the club. Yet Hawking is so enamored of his *bon mot* that he uses it willy-nilly as a book title, confident that most potential purchasers will know little or nothing of the real Kronecker. Meanwhile, far from sounding reactionary, Kronecker sounds *to me* like the voice of reason, by the way:

> As nothing less than the whole edifice from Eudoxus to Cantor is at stake, little wonder that these views [of Kronecker's] cause a stir in the mathematical world. 'Of what use,' said Kronecker to Lindemann, 'is your beautiful investigation regarding π? *Why study such problems, since irrational numbers are non-existent?*' So back we are once more at a logical scandal such as troubled the Greeks. The Greeks survived and conquered it, and so shall we. At any rate, it is all a sign of the eternal freshness of mathematics.
> —Turnbull, 1951 [1929], I:168, italics added. (The allusion is to Lindemann's 1882 demonstration of π's transcendence.)

Most of the issues raised in the present article turn out to be distinctly Kroneckeresque, but that is beside the point here.

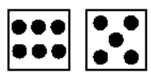
Given all the double-think and shenanigans (such as Hawking's shell game about God and the Integers as described above), how can we break through to the numeration system itself and clear the air? For that purpose, let's turn now to a thought experiment about the number eleven. (For this thought experiment, I take my inspiration from Rahman 2014:66.) Think of 'eleven', please, then match one of the items in Figure 5 to your mental image of it.

Items [a] through [f] are some of the possible responses to "Think of eleven," varied by occupation (or preoccupation); psychology; temperament; nationality; and so on. There is no 'wrong answer'; sadly for us simians, neither is there a right answer, for all of them look equally awkward and irrelevant against a cosmic backdrop.

[a] 11

[b] B

[c] 1011

[d] 

[e] ●●●●● ●●●●● ●

[f] 'eleven' (or '*unsprezece*' in Romanian, etc.)

The exercise was inspired by "Think of 'elephant,' think of 'fifteen' " in Rahman 2014:66 (which in turn might be an allusion to "I can't hold 15 in my hand" in Clegg 2003:152?)

Figure 5: What is 'eleven'?

For most of us, the answer will be [a]. A firmware engineer, if working long hours recently, might choose [b], where we show eleven in hexadecimal, or conceivably [c], where we express eleven in binary notation. Similarly, after a busy night at the casino a gambler might plausibly envision [d]. Someone on the autism spectrum, inclined to think visually about everything, might respond with [e], where my intention is to represent his/her mental picture of, say, marbles, conveniently subgrouped. Finally, to complete the list, a purely literary person (or a smart-aleck) might think 'eleven' (or *unsprezece* in Romanian). Fine. There are no wrong answers — yet.

Now we turn to the real question: Of the half-dozen choices, which would the Creator (or, if you prefer, an advanced extraterrestrial) point to as *the* number eleven itself? On reflection, we see that none of our answers fits the bill as *the* number eleven itself. The base-10 notion that undergirds '11, 12, 13...' is the legacy of our ancestors who bequeathed us two thumbs and eight fingers, echoed by ten toes. No gift from *der liebe Gott* there, only branch-grabbing happenstance. Nor can 'B' lay claim to being the actual eleven, given the abstruse nature and historically late formulation of its hexadecimal base. Ditto for binary. As for marbles in a queue, intuitively one would argue that such is not abstract *enough* to be a serious contender. (Cf. Jourdain pp. 21 and 35 on 'symbols of number.') Likewise 'eleven.' Candidly, we haven't a clue how the genuine number eleven should be represented even here on earth, never mind on a higher plane, or in God's House.

Taking it a step further: In the cosmic eye, is there even such a thing as the counting numbers (1, 2, 3...) by whatever names/symbols/axiomatic scheme? True, we can readily point to *indirect* demonstrations that we are 'thinking correctly,' such as 'making an atomic bomb' or 'navigating to the moon,' but where is the *direct* confirmation that our counting numbers are real, not just an artifact of the culture on monkey-planet? Suppose we cite the composition of the chemical elements, with 1, 2, 3 protons defining hydrogen, helium, lithium...? There I believe we might have our first (and only?) sign of encouragement, reflected back from the universe itself.

Concluding Remarks

Getting real: "But like a diseased financial giant, isn't the Institution of Pi 'too big to fail'?" Point taken. Accordingly, I propose that the scalpel be wielded carefully, as follows: In physics and engineering, the symbol 'π' should stay since it causes no harm there, in its role as a practical tool. The place where its use should be discouraged is, ironically, within the citadel of Queen Mathematics herself. There, to counteract 20-odd centuries of crypto-mysticism and double-think, π should be replaced, in the vast majority of situations, by '3.14' or '3.1415' or 'the 3.1415-algorithm' or 'the 3.1415 algorithm family.' And similarly for ϕ, $\sqrt{2}$ and *e*. After all, 3.1415, 1.6180, 1.4142 and 2.7182 all have strong personalities and are quite capable of 'announcing themselves' on the instant, without the aid of religious iconography.

Similarly, one might argue that '∞' as used in physics (e.g., to define the maximum energy level for an electron, opposite of its ground state energy) is a useful tool. (Note the term 'pure mathematics' in the Nagel/Newman passage quoted in the Prologue.) But elsewhere, there are numerous situations where traditional 'lim n→∞' should be replaced by 'algorithm, for Ǝ' or 'algorithm, for क ' (borrowing the Devanagari letter *ka* that occurs in *kalp* 'kalpa' and *anant kaal* 'eternity'). That way, the distinction between an eternally living asymptotic curve and its dead abstract limit on the whiteboard is not smeared about for the simian convenience of 'getting an answer' or 'breaking for lunch.'

In sorting this out, it helps also to make a three-way distinction between *tags*, *symbols* and *names*. The term 'mileage' is understood as a useful *tag*, nothing more or less. I.e., no one believes that 'the mileage on a Honda' denotes an actual *thing*. The symbol π should be used the same way, and so it is in physics and engineering, where it plays the role of pragmatic *tag*. But in mathematics, the symbol π has become more like the name 'Santa Claus' where some who hear the *name* mistake it for evidence of the existence of a personage or a *thing*.

Above I have made engineering out to be one of the 'good guys' but in fairness we should note also the following negative role that engineering has played in the story:

> Far from pleasing the faculty councils at the École, Cauchy's *Calcul Infinitésimal* incurred their wrath for being too theoretical and not sufficiently practical. In fact, in late 1823, the Minister of the Interior appointed a commission including Laplace and Poisson to ensure that the instruction in mathematics was attuned to the needs of engineering students. For the rest of the decade the École's administration continually monitored Cauchy's lectures to guarantee their suitability for engineering students. —Hawking 2005:641

This provides an important clue as to why certain concepts in the area of limits and asymptotes were gradually degraded in mathematics, so that Cauchy's 'converge toward' was supplanted by 'converge to', the flea-hop, etc., as described above. It seems likely that continuing pressure from Engineering departments, not just in France in the 1800s but everywhere for the ensuing two centuries, drove those changes.

Note that many parts of conventional mathematics remain untouched by the 'assault' above on some sacred cows. Left standing, as it were, are many cherished parts of the landscape, including the ζ-function (see de Sautoy, *The Music of the Primes*); *i* (see Nahin); the Mandelbrot set; bifurcation nodes as they relate to Feigenbaum's 4.669201 (see reference above to Ekeland 1988, Appendix 2); calculus as the celebration of curves (Boyce 2013, Chapters III-IV); and, as discussed already in connection with Ramanujan, the *right-hand* side of the so-called 'π equation' of your choice.

On the long-term prospects for our 'Queen of the Sciences' (Newman I:294, quoting Gauss in praise of mathematics). Once upon a time, the word *alchemy* simply meant 'chemistry' (from Greek *khēmia*, 'the art of transmuting metals,' by way of Arabic *alkīmiyā*). Do present-day chemists look down upon the alchemists of yore? Not exactly. Theirs is accepted as an important protoscientific tradition. Still, some of their specific pursuits are frowned upon, even ridiculed, e.g., taking quicksilver (Hg) as an elixir (presumably because 'quick' means 'living'). And in forming an overall impression of alchemy today, it is difficult to avoid the words *quaint* and *wrongheaded*, somewhere in the mix. I predict that present-day mathematics, if it does not clean up its act, will suffer a fate similar to that of alchemy. A few centuries hence, it will be abandoned in favor of something else (computer science is a good candidate), and remembered only as an important phase in our simian development, now too quaint to warrant much attention, except by historians.

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