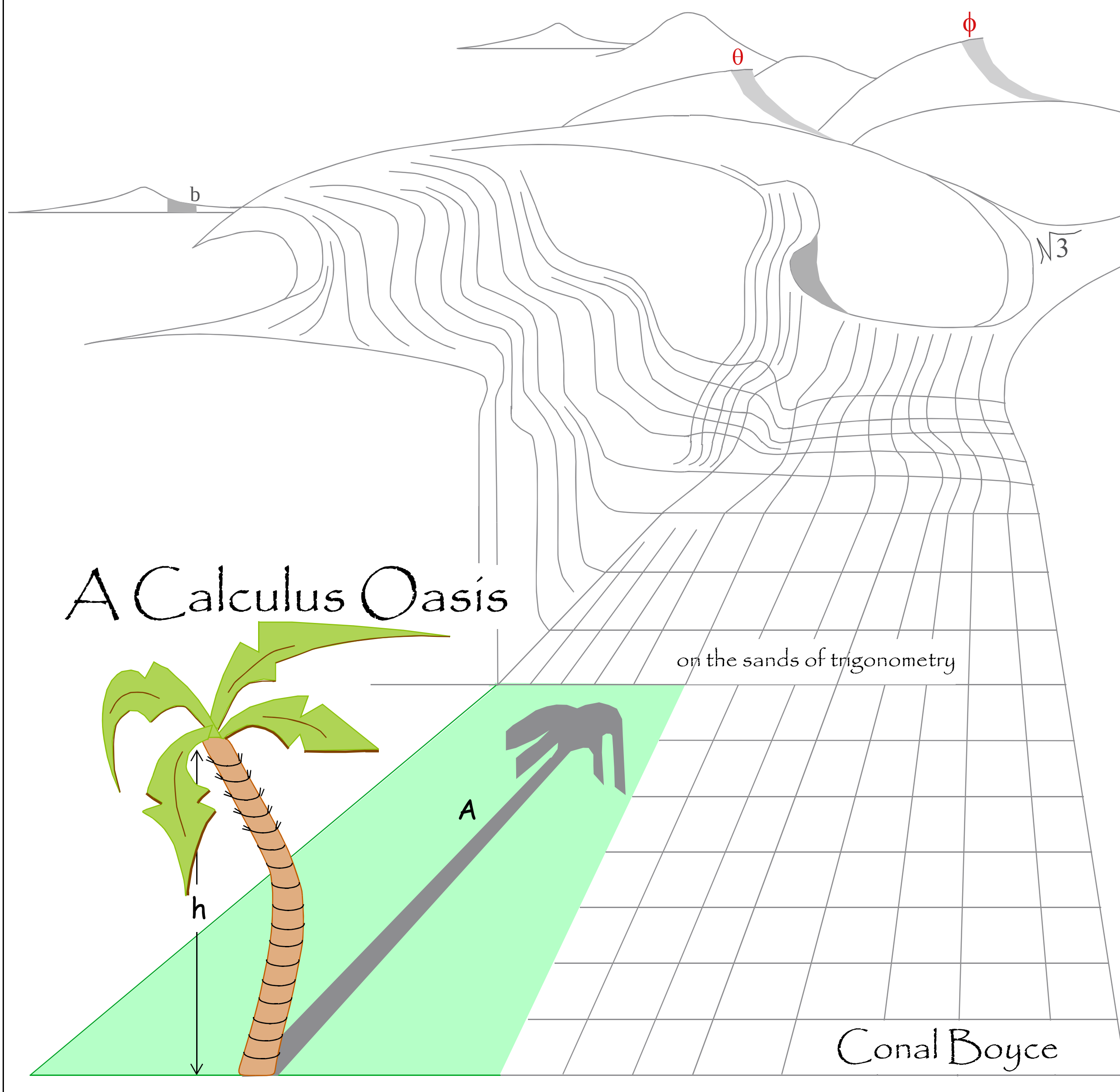
A Calculus Oasis

on the sands of trigonometry



Conal Boyce

A Calculus Oasis

on the sands of trigonometry

with 86 illustrations by the author

Conal Boyce

Cover design and layout by CB

By the same author:

The Chemistry Redemption and the Next Copernican Shift

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In memory of Dr. Lorraine (Rani) Schwartz, Assistant Professor in the Department of Mathematics from 1960-1965, University of British Columbia

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Prologue

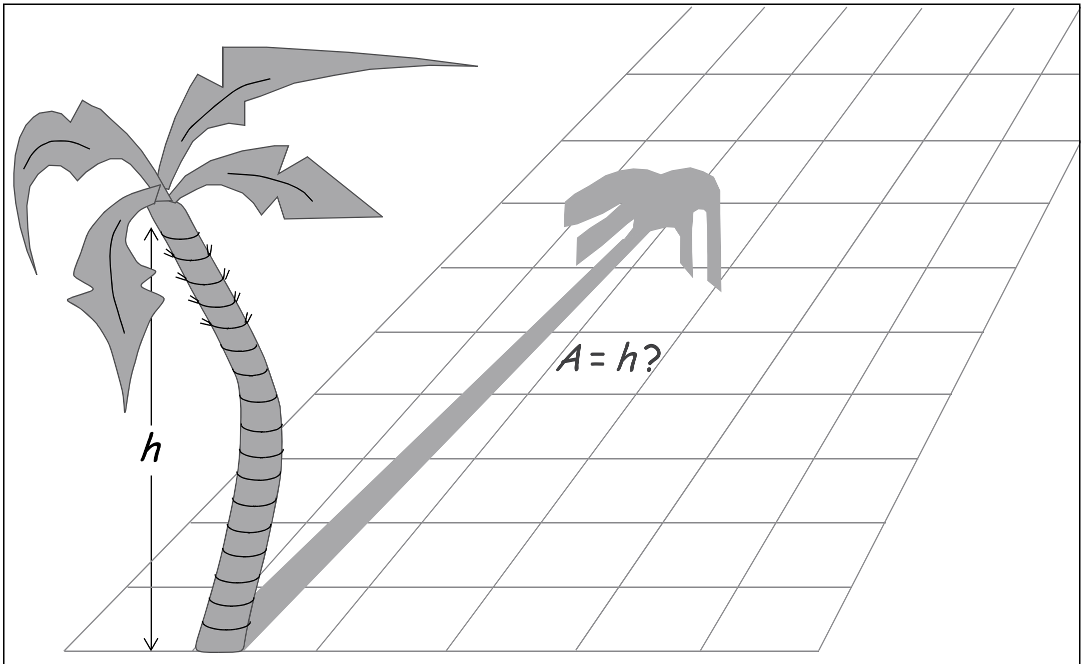

FIGURE 1: A Palm Tree and Its Shadow at 8:17 a.m.

To the question posed in Figure 1, most of us would respond to this effect:
"Why even consider such silliness? It smacks of magical thinking. Moreover, it implies such a mishmash of dimensions (confounding length with area) that one is inclined to dismiss it out of hand. It's an affront to the Church of Science." And yet, in calculus there exists an ironclad two-way relation that is just as incredible-sounding, this time *not* just a traveler's mirage. This (truly) enchanted

relation, the essence of calculus, is invoked simply by appending, to any letter or number, a prime symbol. Thus, f becomes f' or V becomes V′ or 6 becomes (6)′ (as on page 78). The prime symbol (′) is shorthand for *'The derivative of...'*. In Figure 2, I show one whimsical and one real example of the relation.

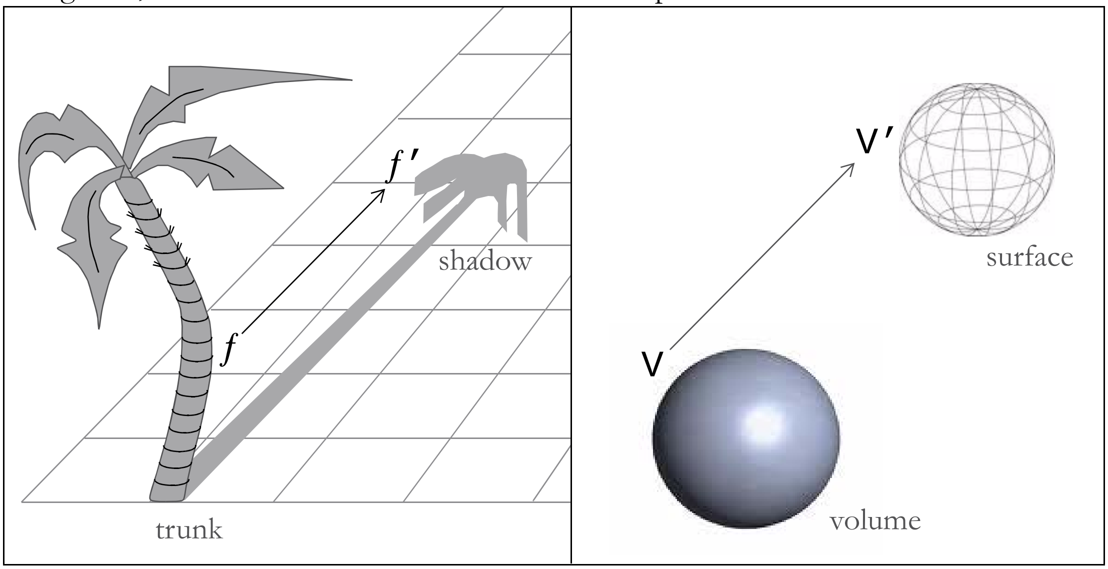

FIGURE 2: Palm Tree and Shadow, Ball and Sphere

How difficult was that? You already know some calculus. Well, some of the *what*, at least, if not the *how* or *why*. (For instance, I've introduced an asymmetry that some may find disconcerting: Going from height to area we 'step up' from 1D to 2D, but in the particular case of a ball and its spherical surface, the derivative involves a step *down* from 3D to 2D, not a step up to 4D as it 'should'; this expectation is reasonable, and it will be revisited and satisfied much later by a different 3D object.)

But before pursuing all of that, let's take time to appreciate a few aspects of the notation system that are more *aesthetic* than technical. As you may recall from middle school math, the equation for the volume of the spherical ball in Figure 2 is $V_{ball} = (4/3)\pi r^3$. Among the half dozen ways to denote a function's derivative (as defined on pages 27 and 207), let's look at two that entail the simple act of appending the prime symbol to a piece of the original equation. As it happens, the derivative of the V_{ball} function is $4\pi r^2$. (Never mind why; we'll obtain the short expression with r^2 from the longish one with r^3 by a simple arithmetic trick in **Chapter VI**.) Accordingly, if we apply the prime symbol this way, to the left side of the above equation...

$$V'$$

...we have created a convenient shorthand for $4\pi r^2$. Next, suppose we apply the prime symbol to the whole right side of the volume equation instead:

$$[\,(4/3)\pi r^3)\,]'$$

That expression *too* must be a synonym for $4\pi r^2$, since once again it says:

Take the derivative of $(4/3)\pi r^3$

Another quick example, closely related to the one depicted in Figure **2**: If the area of a circle is A, then its circumference line may be expressed as A'.

Without seeing prime notation in context, it is difficult to imagine its full power and utility. One can obtain a preview now by perusing **Chain Rule: Single Variable** on page **83**.

Since the prime tick denotes the first derivative, multiple ticks can be used to denote the second derivative (f'') and third derivative (f''') — but thereafter superscripted numerals are used instead: $f^{(4)}$, $f^{(5)}$, and so on. Elsewhere, in Appendices **F** and **G**, we will have much more to say (generally not very nice things) about certain notation practices of the calculus establishment. For now, let's savor the power and elegance[1] of Lagrange's prime notation. Never mind calculus for the moment. It is simply *the* best notation device ever, past present or future. (If you are curious to learn more about the V_{ball}-to-V' relation specifically, see pages **125** and **148**.)

Vintage Calculus versus Wonk Calculus

[*If you are interested to know what* **kind** *of calculus you have been studying (or are about to learn), this is the place for you. If the issue doesn't even sound meaningful to you, feel free to skip this section.*]

The Ritual: The college calculus textbook begins with a slow and meticulous rehash of certain precalculus topics — as though in mortal fear that even breathing the word *calculus* prematurely might jinx The Ritual (represented by the mandala in Figure **5** on page **13**). Given such a ponderous and circumspect approach to a topic, isn't one entitled to believe the topic must be something real, a dignified member of the academic community at least? No. It turns out that college calculus, for all of that elaborate delicate pedagogical megastructure is just a throwaway joke to the math major or 'wonk' (page 227). After all, it's 'only' calculus-*for-engineers*. The real thing is to be approached by way of an even weightier ritual: "Let's do a brain wipe on everything we were taught by our high school or college calculus instructors, and start over with real analysis and symplectic topology and manifolds-with-corners and the differential forms of Élie Cartan and... Now we're running in the tall grass with the big dogs!" Subject to the usual one or two howls of minority-view protest or a qualifying footnote, the picture I've just painted is essentially the truth.[2] Once having glimpsed the lay of the land, how do we vintage calculus enthusiasts find the courage to proceed?

Here is my rationale, in part. For the vast majority of students, vintage calculus (alias college calculus alias calculus for engineers) provides a perfectly good window on mathematics, a glimpse of a world far beyond algebra. After all, it even provides an inkling of what those slackers Leibniz and Newton spent their time thinking about, so how bad can it be? From the perspective of this hypothetical 'reasonable student', the issue about a right and wrong *kind* of calculus is not likely to incite much reaction. Even knowing that the 'correct' path is the one through topology and manifolds, my hypothetical student will still choose the 'wrong' path, the one marked by signs that say clearly Calculus I, Calculus II, Calculus III. There are many possible reasons for the choice: Perhaps one works around engineers and is curious to know what this part of their education was like. Perhaps the path to wonk calculus looks too long and arduous; God knows even vintage calculus is hardly reputed to be an *easy* subject. And so on. Having thus acknowledged the vintage/wonk distinction, from here on I will use the term 'calculus' to allude usually to the vintage calculus milieu with all its special problems, not to the wonk calculus milieu.[3]

Back to business: Given the crushing weight of The Ritual as described above, it is fair to ask how much *any* author (with or without my particular bias which is pro geometry and pro pictures) dares to depart from the script. The answer is: not by much!

When I think of the extreme delicacy of h in the difference quotient, to be described on page 27 below, or the façile cleverness surrounding the constant rule and '+C', as described on page 78, it does indeed seem like a house of cards plainly in need of mathematical rigor to keep it propped up. But when I look at the stark beauty of the FTC as in **Figure 24 (An Antiderivative (aka the Original Function) Casting**

Its 'Shadow') on page **45**, or the lyrically beautiful geometry of derivatives, as distilled to **Figure 25 (Kartouche with 'Slide Rule' to its Left and Illustrative Curves to its Right)** on page **48**, I feel it is silly to tiptoe around calculus as a subject. It is robust and requires no such pussyfooting. Thus, one of the motivations for writing this book: to celebrate the beautiful robust side of calculus. But a whole *chapter* (**Chapter IV**) devoted to curves? There are two reasons for this, beyond the aesthetic appeal:

[1] In celebrating the curves we are trying to address the '80/20' problem. Here I allude to the following conventional wisdom: "Getting through a calculus class involves about *80 percent algebra and only 20 percent calculus.*"[4] To see how this 80/20 ratio plays out in some extended examples, please refer to **Appendix A**.

[2] I believe the curves of calculus pass the Little Green Aliens Should Not Laugh test. Imagine a visit to earth by the proverbial little green aliens. Do we have any science or technology that is likely to impress them? Rather than cast about for some math or science that might impress such envoys from deep space, humans should consider a more modest goal: Just please don't make the visitors *laugh* at our methods or artifacts, for being so provincial and quaint. By my lights, only two things pass that test: As your little green visitor with polished onyx eyes, I won't laugh at your earthling chemistry, and I won't laugh at the curves of your earthling calculus. But for all the rest (higher math, physics, and 'the Age of Information'), beware. I might never *stop* laughing. An easy way to appreciate the universal quality of the curves of calculus is to think about them in terms of vintage calculus versus wonk calculus: From the viewpoint of the former, the latter might very well have come 'from space aliens'; and yet, when the curves are drawn, they look the same in both cultures (let's hope!) This thread continues with the discussion of Figure **31** on page **52**.

In its original conception, this book would have contained only the material now seen in Chapters **III**, **IV**, **V** and **VII** and Appendices **A**, **D**, and **E**. But I soon came to my senses and added Chapters **I**, **II**, and **VI** and Appendices **B**, **C**, **F**, and **G** as practical foundation stones, on which to erect the more personal enthusiasms and diatribes. Glancing only at Chapter **I**, **II**, or **VI**, I believe one will see 'a conventional book about calculus'. But when one takes into account the *core* chapters and appendices as enumerated above, the overall picture is that of an offbeat treatment. (For the record, the working title of this book in the earliest of its two or three phases was *Calculus as the Language of Curves*, with subtitle *Paying Due Respect to the Ethereal 20% of the Curriculum that Dances Free of the Heavy Chains of Algebra*. Over time, competing themes emerged, e.g., the FTC Geometry of **Chapter VII**. So the title had to be changed.)

A note about the so-called 'appendices'. In math books, we are accustomed to seeing phrases such as the following: "A review of determinants is presented in Appendix 3." My appendices are not like that. They contain exposition — it just happens to be exposition that I thought might interrupt the flow of the narrative (such as it is), that's all. Although I dislike the name and the sound and the connotations of 'appendix' I can never think of a synonym to use instead. Long story short, read the appendices! In my books, they are as important or more important than the 'chapters'.

Assumed Audience

Outside of the confines of the calculus textbook proper, there is a whole world of auxiliary books that might be classified along the following lines:

[a] the *instead*-of-calculus books

[b] the *relief*-from-calculus books

[c] the *supplements* to a specific or generic calculus curriculum

Berlinski's *A Tour of the Calculus* exemplifies type **[a]**. Its subtext is: "You probably haven't taken calculus yet, and you probably never will, *but* you're still curious about it since it seems to be something with which every educated person should at least have a passing acquaintance. Here is a book that will take you to the inner sanctum and give you a glimpse of Valhalla, painlessly, as literature!" (On balance, it works, although I object to his 'black jewel of calculus'; see **Appendix E**. For a caveat regarding type **[a]** generally, see the discussion of Figure **5** on page **13**. Also, don't forget the higher-level caveat implied by the discussion of vintage versus wonk calculus on page **3** above.)

Every now and then a book will come out with a title such as *The Manga Guide to Calculus*. These comic book approaches define type **[b]**, the *relief* from calculus books. One imagines such books being used as a kind of dormitory R & R, as comic relief from the daily grind of a mainstream class, although some bookstore browsers may peer wistfully at them in foolish hopes of having found a type **[a]** book, we suspect.

Mark Ryan's *Calculus Workbook for Dummies* is a high-quality example of type **[c]**, the companion book or *supplement* to a calculus curriculum. Likewise St. Andre's *Study Guide* and Schey's *Div, Grad, Curl, and All That: an Informal Text on Vector Calculus.* (Schey's book has the look of a 'classic' perhaps? If it is such, it is certainly not an easy classic, though it is packed with insights that are generally absent from textbooks, e.g., his half-dozen pages devoted to the physical meaning of *curl*; Schey, pp. 86-91.)

But soon enough we encounter books where my proposed taxonomy doesn't work very well.

Amdahl & Loats' *Calculus for Cats* partakes somewhat of all three categories. Per its self-description on p. 9, it belongs primarily to type **[c]**, but the authors allow that it might also work for some readers in the role I call '*instead*-of-calculus' or type **[a]**. Finally, by its whimsied connection with cats it exhibits some type **[b]** qualities as well (and was once given to me in that role by one of my daughters as I followed in her footsteps, starting my calculus education at age 60). However, far from using drawings and cartoons to enhance the presentation it addresses itself to a readership (large?) who are soothed by seeing *more* words, *fewer* graphs and symbols — just the opposite of my approach here under a premise of the more pictures the merrier.

Priestley's *Calculus: A Liberal Art* is likewise a mixed bag, although in a very different way. It *tries* to be a stand-alone volume that can be traversed in the instead-of-calculus mode, *à la* Berlinski; at the same time, it contains many examples and exercises that have the look-and-feel of a humdrum textbook, plus an attempt at some nonstandard 'rules' on pp. 155 and 164-169 that are only an unwelcome distraction from the standard rules one *must* learn.

So much for the context. What about the present volume? From its title, *A Calculus Oasis*, it would seem to be type **[b]**. But really, like *Calculus for Cats*, the present volume doesn't fit comfortably into any of the categories I've proposed, touching instead on all three of them, roughly in the diminishing proportions suggested by Figure **3**.

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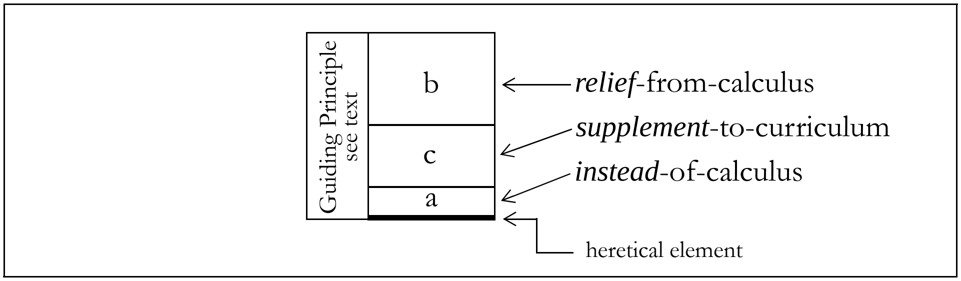


FIGURE 3: Content Self-Analysis (Rough Estimate)

Thus, as with *Calculus: A Liberal Art*, there is a danger that the reader may be put off by my book for being neither fish nor fowl. In my defense, I'll remind you that there *is*, however, a guiding principle already alluded to: namely, to move the '20% calculus' (however fleetingly) into the limelight.

And what of the other 80%? On the cover, I've characterized it as 'the sands of trigonometry', illustrated by my naive sketch of a desert. Am I thus suggesting something derogatory about trigonometry? Not really. One must accept trig as a necessary tool (or 'necessary evil') which, in its own way, can even be beautiful at times. (Hence my attempt at some lyrical curves on the far horizon of the cover art.) In fact, **Appendix C** (as supplement to Tables **4** and **5** in **Chapter VI**) is my tacit acknowledgement of the critical importance of trigonometry. So don't be fooled by the subtitle: Yes, this book attempts to be a celebration of pure calculus as a kind of 'water at the oasis', but it is *not* therefore a (total) escape from or disparagement of trigonometry — the metaphorical dunes that are never more than a few yards distant.

Another reason it is difficult to classify this book as type **[a]**, **[b]**, or **[c]** is that it includes criticism of the status quo. The thin black rectangle lurking at the bottom of Figure **3** represents this dash of heretical pepper. These materials I include because they may provide moral support to other students who have been bothered by the calculus pedagogical machine, whose entrenched customs run the gamut from the quaint to the downright stupid. Or bothered by certain holes in the curriculum — holes that are no one's fault in particular, but gaping nonetheless. At the same time, I realize the student is too busy working with the existing symbols and nomenclature to have any appetite for actually trying out new ones, so I keep most of my comments within in the confines of Appendices **D**, **E**, and **G**. With the 'heretical' layer, it is not that I am trying to win converts, merely air a few ideas, something different for the student to consider along the way. (Well, inside the Establishment, I might be hoping to win say, *one* convert at least. But here I am thinking about the reader who is a student of calculus, down in the trenches, where one has little patience for such a 'political' battle. Which, in any case, might be better characterized, at times, as me tilting at windmills, due to the context outlined on page **3**.)

Scope

Elsewhere I've made a few comments on the fine structure, so to speak: which ingredients went into certain chapters, and why. Here I will comment instead on the book's gross structure and scope. From its size (reasonably large) and from the general appearance of some of the chapter subheadings (Integration by Substitution, Integration by Parts...) it may seem that I intend it as a comprehensive sweep through all the major topics of elementary calculus. Its range of topics *is* wide, hitting say 70% of the key topics in Calculus I, II and III, but this is emphatically not a textbook. Next question: Assuming my 70% guesstimate is in the ballpark, then *which* 70% of Calculus I, II and III gets covered here? Have I merely cherry-picked a few pet topics? No. I've fulfilled my pledge of an 'oasis' in two very different ways: (a) by emphasizing *some* favorite topics, I admit (e.g., in **Chapter III**); (b) by going deeply into some thorny topics too (e.g., in **Chapter VII** and **Appendix D**) in ways you won't find elsewhere. So the book is serious — perhaps too serious depending where you first happen to open it.

Calculus III

Notice that the book includes significant Calculus III (vector calculus) content. This sets it apart immediately from the vast majority of companion books which, even when they call themselves 'humongous' and run to 1000+ pages of 'solved problems', almost always confine themselves to the realm of Calculus I and Calculus II, implicitly, their authors feeling no compulsion to offer even the first shadow of a hint of what might lie ahead in Calculus III.

What the traditional first-year curriculum *does* include is a sneak preview of Calculus IV, by way of a quick unit on differential equations, tossed in as a sort of coda at the end of the year, and that practice in turn is reflected in many of the ancillary books outside of academia. But this only makes the silence on Calculus III topics that much more conspicuous and puzzling. Outside of Schey's classic, nothing springs to mind. (At first glance, Bressoud's book may seem to be a third-semester companion book, but it is actually the opposite: a wonk calculus primer; for details, see page **229**.)

While **Chapter VII** is dominated by Calculus III material, I would recommend that a Calculus I/II student give it a quick read-through anyway since it suggests a kind of road map pertinent to all three semesters.

Sources/Influences

The section in Salas and Hille (1990) called 'What Is Calculus?' probably nudged me toward trying an unconventional approach to the subject. Not to say that Salas and Hille indulge in the sketching of palm trees and dunes, but they do rely heavily on graphics to answer their own question. The main answer comes in the following nonverbal form: sixteen pairs of graphics illustrating concepts from elementary mathematics juxtaposed with their calculus counterparts, as represented by a brief excerpt in Figure **4** below.

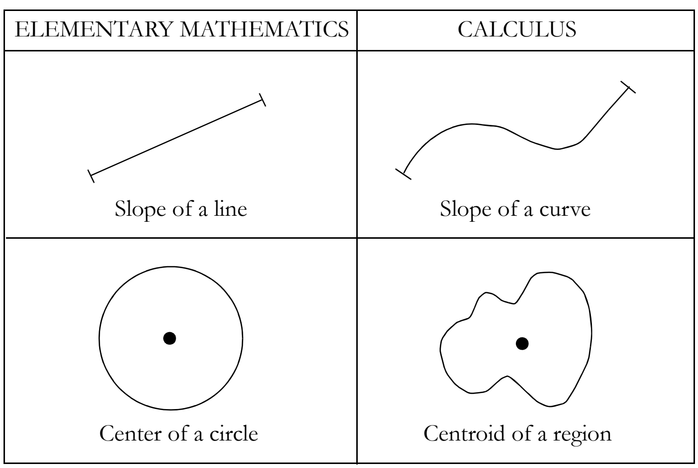


FIGURE 4: What is Calculus? (after Salas & Hille, pp. 1-4)

Meanwhile, in words, they answer the question this way:

> [C]alculus is elementary mathematics (algebra, geometry, trigonometry) enhanced by *the limit process.*
>
> — Salas & Hille, p. 1
>
> (The italics and the modulated font size are in the original.)

I was inspired also by Berlinski's notion of curves as the 'faces' of their respective functions (Berlinski, p. 144n and *passim*).[5] Another influence was Priestley's discussion of $dA/dr = C$ (circumference understood as the derivative of a circle's area; see Priestley's cover or Priestley, p. 221). My earlier criticism of Priestley notwithstanding, he does, at times, succeed in bringing a kind of poetry to the subject, and that aspect of his book I still admire. Robinson *et al.*, pp. 69-70 and 78, are the basis for Figures **31-32** on pages **54-55**, which I conceived of originally as the book's centerpiece (although they haven't quite that prominence in the current incarnation of the text). For other sources, see **Acknowledgements** on page **248**.

A Quick Backward Glance at Precalculus and Related Topics

If we look back at precalculus, it too has a split within its materials, similar to the 80/20 split for calculus.[6] Thus, much of what we thought would prepare us somehow for calculus did no such thing; rather, it prepared us for the 80 percent of the curriculum that is *not* calculus but 'algebra' (still)! All-told, calculus and precalculus constitute a field with a most peculiar set of tribal customs, many of them internalized and no longer even remarked upon.

One who does take the time to survey the landscape and comment on its oddities is George F. Simmons, author of *Precalculus Mathematics in a Nutshell*:

> Geometry is a very beautiful subject whose qualities of elegance, order and certainty have exerted a powerful attraction on the human mind for many centuries...In spite of all this, most high school students emerge from their geometry courses with mixed feelings of confusion and relief. Why?...They have been bombarded with innumerable nit-picking definitions, and also with elaborate, boring 'step-reason, step-reason' proofs of statements that in most cases are obvious to begin with. — Simmons, p. 2

> I make no effort here to con the student into believing that algebra is useful or exciting for everyone in everyday life. It isn't. Its importance lies in the student's future... — Simmons, p. 33

> I have been teaching trigonometry for more than 30 years, as a brief but essential interlude in calculus courses...*I routinely cover everything that matters in trigonometry — from the beginning, with proofs — in a single 50-minute lecture.* Under these circumstances, it was only natural for me to ask myself why this subject consumes an entire semester in most high school curricula...hundreds of pages of unnecessary padding, consisting mostly of obscure formalities and irrelevant digressions...Most trigonometry textbooks have been written by people who appear to believe that the importance of the subject lies in its applications to surveying and navigation...heights of flagpoles...positions of ships at sea. The truth is that the primary importance of trigonometry lies in a completely different direction — in the mathematical description of vibrations, rotations, and periodic phenomena.

— Simmons, p. 92-93 (italics added)

The Simmons passages are well worth reading in full. I've abridged them above only as a precaution to forestall a possible violation of the fair use copyright principle. I highly recommend the body of his book, as well. By design, Simmons' book is very different from a book such as Hungerford's, but it is excellent in its own way, more in the role of reference book than textbook perhaps. From time to time there are rumblings to this effect in the educational establishment: "Why are our math textbooks so enormous, running to a *thousand* pages with myriad color plates whereas a Soviet textbook covering the same material requires fewer than a *hundred* pages, all black and white? (yet those Soviet mathematicians and scientists seem so darned clever!) Shouldn't we *do* something?" Of course nothing ever changes. But Simmons' book when compared to Hungerford's — which I also like very much — is a perfect example of what that dreamed-of slimmer textbook *would* look like, if such a revolution were seriously pursued. (I'll admit that 'a single 50-minute lecture' for trigonometry sounds absurdly short. To make sense of that passage, consider Figure **71** on page **164** below. If I were teaching precalculus, I might spend ten minutes presenting that diagram, but to actually learn it, a student would in turn spend a much greater amount of 'quality time' with it (as the joke goes).

A Coffee/Calculus Analogy

Perhaps what you want is 'a cup of coffee', as in a 1940s film? Dream on. Nowadays you must engage the whole coffee-merchandising megamachine to obtain one. Possibly you forget to enjoy your 'hot beverage' after such a harrowing experience. The situation bears some resemblance to the dilemma of 'learning calculus'. There is no straight smooth path to the subject; there are innumerable distractions along the way. In Figure **5**, we compare and contrast the two situations.

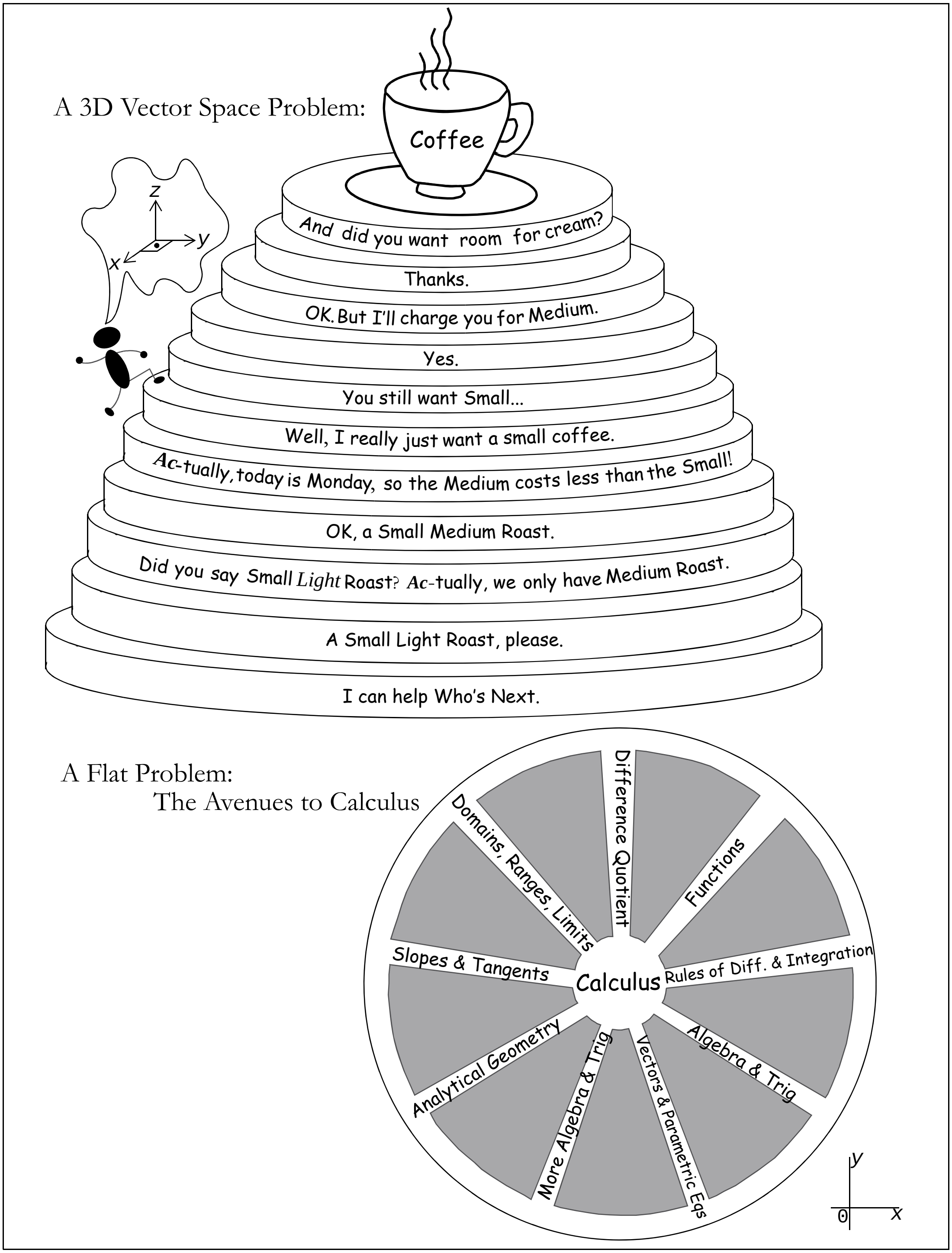


FIGURE 5: Toward a Proof of the Coffee/Calculus Theorem

All the spokes need to be there for the calculus 'wheel' to come alive, but for several of them there is no particular sequence that is required. Rather, one needs to approach the center repeatedly down multiple avenues (to change the metaphor yet again). Slowly but surely that one thing, calculus, emerges from the fog.

So, which process is more difficult — learning calculus or obtaining a cup of coffee? By inspection of Figure 5, it should be clear that we have cobbled together our 'proof' already of the following Theorem:

> While calculus is *reasonably* difficult, its difficulty (modeled as a flat mandala) *pales* beside the challenge (the Rings of an Inverted Inferno) of obtaining a cup of coffee in the year 2011.

Note in passing that Caribou engages in the practice of overloaded parameters, with 'Medium' denoting either Medium Roast or Medium Size, dependent on context. What better preparation for the several overloaded symbols used in calculus, as summarized on page **192**? So not to worry: If you've figured out how to survive the Monday Special at Caribou Coffee, you can probably sail through calculus. (Getting Calculus << Getting Coffee, Q.E.D.)

OK, we've had our little moment of fun with the American brand of *barista* culture. Returning to a more serious mood, it seems reasonable to pose the following question: If calculus is really such a tough nut to crack, as suggested by the nine-spokes model in Figure 5, how can a type **[a]** book claim any validity? (the kind of book that seems to offer a direct line ~~to God~~, I mean to calculus) The answer is straightforward: Any book that presents itself as type **[a]** (or that claims to contain type **[a]** elements, as this one admittedly does) is immediately suspect. 'Buyer beware!' 'If it sounds too good to be true, it probably is.' The author's effort may be in good faith (clearly Berlinski's is), but the undeniable 'spokiness' of the standard precalculus/calculus curriculum should give one pause on a suspicion of fool's gold.

Keeping the Goal in Sight: the FTC

There exists a clean simple idea that stands at the very core of the subject. That central *idea* is inherently pretty but it goes by a *name* that contends for title of Most Ugly and Flat-Footed in the Language, namely:

The Fundamental Theorem of Calculus (or FTC)

Calculus is such an intrinsically pretty subject, it is a pity that it can be approached only through a thicket of terms which must rank among the ugliest in the whole vast lexicon of the English language: *FTC*, *logarithmic function, inverse trigonometric function*, *slope, nabla, parametric equation, exponentiation* (which is actually a lot of fun, by the way), *difference quotient, arctan, antidifferentiation, antiderivative, concave up, improper integral, the Second Derivative of the Nephroid of Freeth*... (Just in case you have formed an unnatural attraction to any of the terms listed above [love at first sight?], not to worry: They can all be visited at your leisure in **Appendix G**.) Over at the other extreme, we have *Green's Theorem*, a refreshing scrap of color to brighten the moonscape of linguistic horrors. Otherwise, I fear it would be one long race to the bottom of butt-ugly.

Not only is 'FTC' a hideous term but an academic author will typically postpone even talking about the FTC until the midpoint of his/her textbook. By contrast, I have presented a notion of the FTC early on, albeit in a form that is whimsical: Figures **1** and **2**. Let's develop the idea a bit further. Given a certain 'shadow', we pretend that one may deduce its 'tree', and one's knowledge of the tree can then be used to answer questions about the shadow that would otherwise be intractable or impossible to answer. (Much later, in **Chapter VII**, we will come to realize that this seemingly one-way relation between tree and shadow is frequently exploited as a two-way relation where the higher dimensional FTC variants are concerned.)

If we translate the typical calculus curriculum to our metaphorical terms, then Calculus I is concerned with learning how to make a palm tree (or function f) cast its shadow (or first derivative f') and Calculus II is concerned with tracing the process back the other way, inferring a tree from its shadow then using the tree to answer difficult questions about its shadow. So far so good. Explained only to this extent, calculus may remind one of any number of familiar processes where one thing is converted into another, then back — say U.S. dollars converted into Japanese yen, then back again, with one currency implying the other at all times.

But now comes our first brush with entrenched culture and with symbols that are purely arbitrary (and sometimes just goofy and/or exasperating): In the second semester context, the palm tree is now labeled F, not f, and the shadow is now symbolized by f, not f'. Strange but true. Refer to Figure **6**.

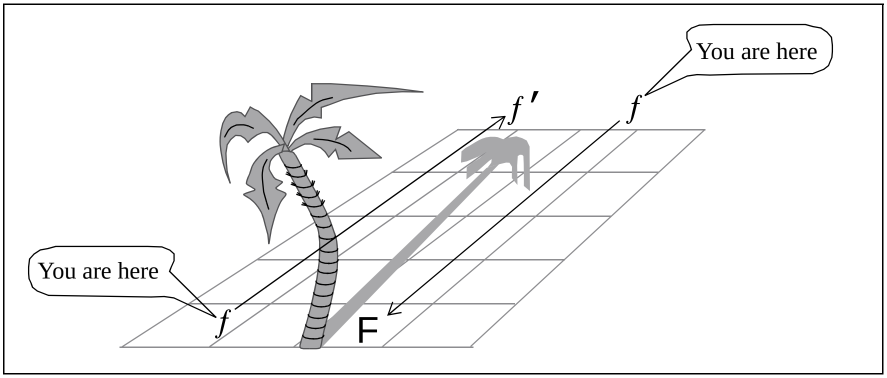


FIGURE 6: The Fourfold F and 'You Are Here'

Thus, to get comfortable with Figure 6, there is a bit of culture shock to overcome. The secret ingredient that holds it all together is that f to a calculus insider, always connotes 'You Are Here' (as explained further in connection with Figure 25 on page 48, also Figure 20 on page 40). In a sense, the whole thing is as simple (or obscure!) as that.

Nomenclature

By now we recognize f' as the derivative of f, but what would be a reasonable name for F in Figure 6? Having been indoctrinated (if not brainwashed) by the pedagogical establishment, I'm not sure what a *reasonable* name would be, but the rather strange one that we're all stuck with forever is 'antiderivative'. (To demystify the term, please refer to page 203. For a preview of how the tree-and-shadow metaphor plays out with an actual function, see Figures 20, 22 and 24.)

Viewed in isolation, Figure 6 may seem a bit contrived or pointless, but after you have encountered it a second time as Figure 22, now heavily annotated, I think it will start to appreciate my reasons for drawing it. As with so many things in calculus, one third of the difficulty is inherent in the material, while the other two thirds of the difficulty are with the notation 'system', or rather the lack thereof. Not your fault (or mine!)

I Slopes and Functions

This chapter covers the concept of slope, several definitions of a function and its derivative, also the difference quotient, and tangent equations.

The Concept of Slope: Picturing Division

Is it possible to 'make a picture of division'? Somewhere along the way in traditional K-12 math we learn that a depiction of three pebbles beside three pebbles is a 'picture of addition', and from there one can make a 'picture of subtraction' by removing a trio of pebbles. Later one might be told that multiplication is just repeated addition, hence division is just repeated subtraction: Make a pile of six pebbles, for instance, then remove pebbles in groups of three, noting that two such subtractions are required to make the pile vanish.[7]

In that discussion typically no mention is made of the portrayal of division shown in Figure 7b, which is just as true to the numbers and requires no 'pebbles' to manipulate, only the static sketch of a right triangle.

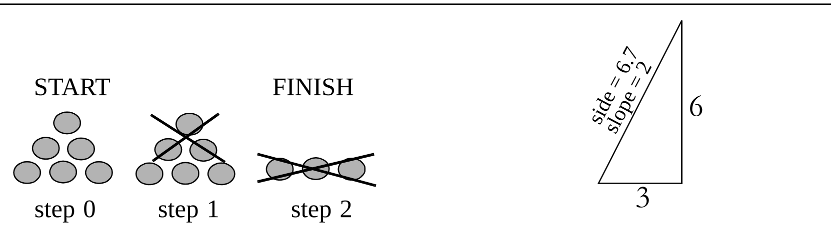


Figure 7a: Six pebbles divided by three

Figure 7b: Rise over run = 6/3 = 2

FIGURE 7: Pebbles and Triangles as 'Pictures of Division'

In other words, every time one carries out a division operation, not only is there an 'answer' (in this case '2') but one is also declaring or discovering a *ratio.*

Specifically, it is the rise-over-run ratio that interests us here. The rise-over-run is a ‘slope’ and it is also a ‘picture of division’.

The other ‘answer’ shown in Figure 7b is the length of the hypotenuse ($\sqrt{3^2 + 6^2} \approx 6.7$), which is only incidental to the current discussion.

You can guess by looking that the length of the hypotenuse is approximately seven units. Similarly, with practice, in the calculus or precalculus milieu, you will be able to ‘read’ the slope of a line directly, *without* even considering what the base and height of its implied triangle might be: this / is slope 1, this / is slope 2, this / is slope 3, and so on.

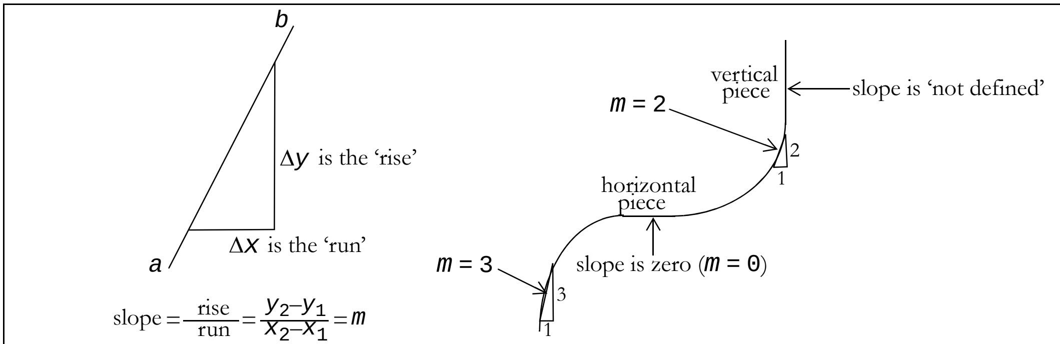


Figure 8a: Generalized Definition of Slope

Figure 8b: Two Special Cases (which also illustrate Base = 1)

FIGURE 8: Definition(s) of Slope

In Figure 8a, we repeat the triangle from Figure 7b, now as the attachment to a curve that runs from a to b (a ‘straight-line curve’, that is). The sides of the triangle now have labels that look more like calculus (or precalculus at least), and we use them to show the generalized definition of slope, which is ‘m equals delta y over delta x’. In Figure 8b, we show two special cases (a horizontal segment and a vertical segment) and we also illustrate the case where a triangle’s base is 1. The latter device is employed frequently to simplify slope calculations. If the base is one, you throw that data away as a nonevent and simply report the y-value *as* the slope, like ‘$m = 2$’ or ‘$m = 3$’ in the illustration. (Similarly, in the unit circle, x/r is represented by a lone ‘x’ because by definition $r = 1$, so $x/r = x/1 = x$; and y/r is likewise represented by a deliciously succinct ‘y’; for a full discussion of the unit circle, please refer to **Appendix C**.)

A slope can be negative as well. For instance, the slope of a sine wave as it passes through $x = \pi$ on the x-axis is negative. (This is a context where I gladly break my own rule about writing 3.14 instead of π; see page **196**.) The slope will carry a minus sign, but to what extent will the slope be negative? We'll consider that next. Because of its connection to the unit circle (**Appendix C**), it is natural to think of the sine wave as a succession of curves that have a rounded, quasi-semicircular look, as shown in the upper part of Figure **9**.

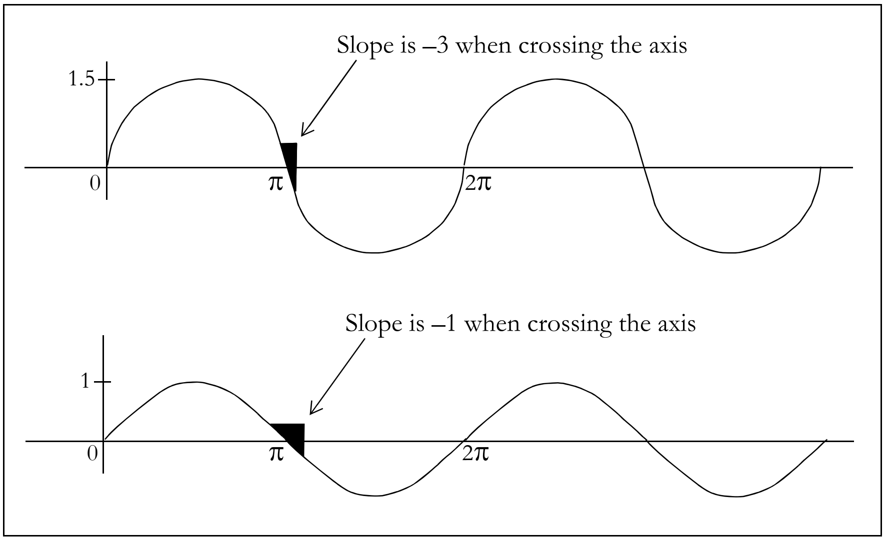


FIGURE 9: Two Notions of the Sine Wave

I've noticed that in a quick, impromptu sketch of the sine wave, the curve at $x = \pi$ is found usually in a steep dive, with a slope of –3 or so. Meanwhile, in the actual sine wave, the slope at that juncture is far shallower: only –1. These two cases are shown together in Figure **9**. Subjectively one might say the conventional version looks more 'natural' than the real thing.[8]

The contrast is reminiscent of an issue with randomness: Humans *think* randomness should look a certain way, but *real* randomness doesn't match expectations. Subjectively, the real thing looks 'not random enough' (see remarks about the iPod Shuffle in Mlodinow, p. 175). In connection with Figure **9**, the corresponding reaction might be that the real curve looks 'not round enough'. The real thing possesses a subtle, nonintuitive shape that is actually rather alien to human sensibilities. Granted, this business about right and wrong ways of representing the sine wave is slightly off-topic (dare we say, 'tangential'?), but it seems as good a way as any of introducing the notion of slopes with negative values. That's the salient point in the context of this chapter.

The Function Defined

At this juncture, as prelude to the difference quotient (page **27**f.), the calculus class veteran expects to see something about the squared function alias parabolic function alias $y = x^2$ (Figure **10**). Probably the reason an author leans strongly toward introducing the parabola at this point is its relatively easy-to-follow table of inputs (x) and outputs (y).

x	y
1	1
2	4
3	9
4	16
5	25

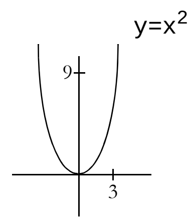


FIGURE 10: Parabola Function

All-told, the function $y = x^2$ offers a bare minimum of distractions for the student, from the subject at hand, which is further development of the slope concept. We too will be exploring the $y = x^2$ function (bringing it back in its usual role in **Chapter V**), but we here will make a 'bold' departure from tradition by shifting our attention to the function whose graph is shown in Figure **11**.

x	y
1	0
2	0.693
3	1.098
4	1.386
5	1.6

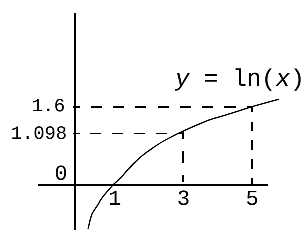


FIGURE 11: Natural Log Function

A natural logarithm is one that takes the number e (2.7182) as its base (instead of using base 10 or base 2). The natural log function, $y = \ln(x)$, may be read aloud as 'Ellen of X'.

My rationale for bringing the natural log function into the picture so early: I think it provides an especially attractive way of demonstrating the Fundamental Theorem of Calculus (FTC). And if it is introduced here in **Chapter I**, then it is available for taking on that role in **Chapter III**, already part of the student's vocabulary when the time comes. Even more important, by looking closely at the natural log function now, we can head off various misunderstandings about functions *generally* later on.

So, how *does* one define a mathematical function? Here is a broad sampling:

> ... a **function** is an association between two or more variables, in which to every value of each of the **independent variables**, or **arguments**, corresponds *exactly one value* of the **dependent variable** in a specified set called the domain of the function. **Map**, **mapping**, **operator**, and **transformation** are other names for a function...It is standard practice to write the **dependent variable** on the left-hand side of the equality sign of an equation; thus, in
>
> $$y = x + 1 \text{ or } f(x) = x + 1,$$
>
> y or $f(x)$ is the dependent variable, x the **independent variable**.
>
> — Gullberg, p. 336-337

> To understand the origin of the concept of function, it may help to consider some real-life situations in which one numerical quantity depends on, corresponds to, or determines another. — Hungerford, p. 125

> When each possible value of x is paired with only one value of y, y is said to be a function of x. — Leff, p. 88

> Let D be a set of real numbers. By a *function* on D we mean a rule that assigns a unique number to each number in D. The set D is called the *domain* of the function. The set of assignments that the function makes...is called the *range* of the function.
>
> — Salas & Hille, p. 26-27

> A function of one variable, generally written $y = f(x)$, is a *rule* which tells us how to associate two numbers x and y; given x, the function tells us how to determine the associated value of y. Thus, for example, if $y = f(x) = x^2 - 2$, then we calculate y by squaring x and then subtracting 2. — Schey, pp. 2-3

> If x and y are two variables that are related in such a way that whenever a permissible numerical value is assigned to x there is determined one and only one corresponding numerical value for y, then y is called a *function of* x.
>
> — Simmons, p. 51

(For each of the above quoted passages, the italics or bolding, if any, are in the original.) Are you as surprised as I was by the sheer variety of definitions? I've listed them all because several contain a 'crucial ingredient' found in none of the others!

Strictly speaking, the Hungerford quote is not a definition, only a prelude to one. Nevertheless, his unassuming sentence holds up best over time, in a wide range of circumstances, *as* a definition; we'll return to it in a moment. The Gullberg definition we shall return to later in a very different context, concerning the Fundamental Theorem of Calculus (page **210**). Except for Hungerford's, notice how all the above definitions, even when they give lip serve to the idea of a general *rule*, ultimately encourage the reader to regard a function as a little input/output *machine*. The latter idea is sometimes illustrated with a graphic such as the 'black box' portion of Figure 12, where I've tried to draw a composite picture covering several of our quoted definitions from above, supplemented by other sources as well to arrive finally at a reasonably complete picture.

A note about the word 'unique': The mathematician's use of the word 'unique' is curiously wanting (i.e., useless to the outsider). I hope that the domain-to-range mapping diagram in Figure 12 will clarify what Salas and Hille mean by 'assigns a unique number' (as quoted above). I.e., as *defined*, a function in the abstract has many-to-1 mapping, not 1-to-1 mapping. However, a *particular* function may happen to possess 1-to-1 mapping, e.g., in the function that maps citizens to social security numbers, and this 'exception' is accommodated by the many-to-one definition. The one thing forbidden is 1-to-*many* mapping. When that kind of mapping occurs, it is no longer a function. All of that freight is carried, implicitly, by the one word 'unique'! Welcome to the wonderfully terse (and sometimes simply asinine) world of math-speak.

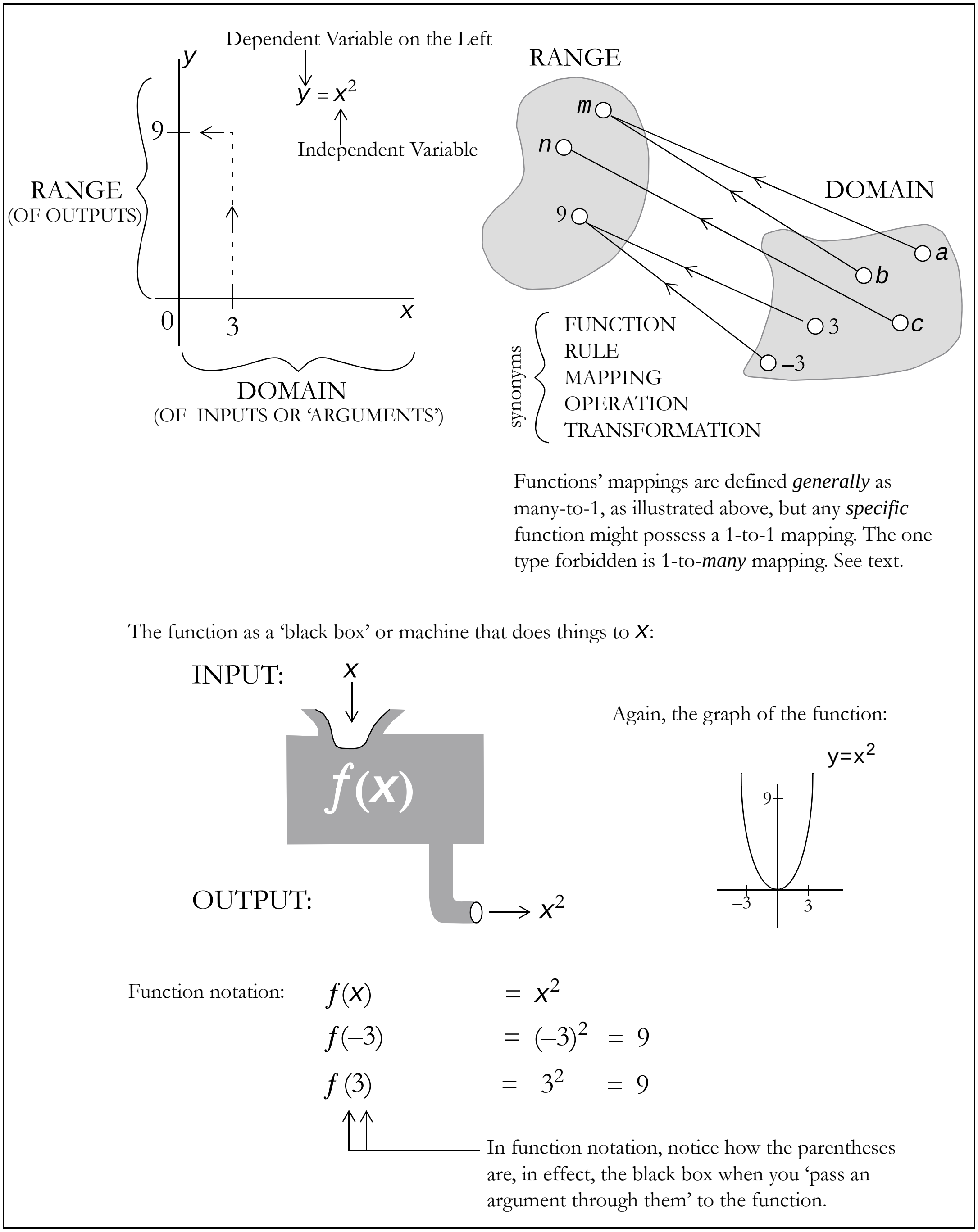


FIGURE 12: The Half-Dozen Ways of Looking at a Function

Well, very often a function *may* be safely regarded as a little machine or 'black box' with cut-and-dried inputs and outputs, as depicted in Figure 12. For instance, with a function such as $y = \sqrt{x^3}$, we don't care *how* the number x is cubed and a square root taken, we just want it *done* somehow, inside the notional black box. But with a function such as $y = \ln(x)$, one had *better* care, and that means developing instead a 'white box' view, to compensate for one's lack of intuition about the input/output relation. Refer to Figure 13. Borrowing Hungerford's terms, $y = \ln(x)$ is a case where one does not want to say y *depends* on x or x *determines* y; rather, it is a case of y *corresponds* to x, by threads of logic that can only strike the human observer as delicate and convoluted, nothing like the simple meat-grinder analogy suggested by the image in Figure 12.

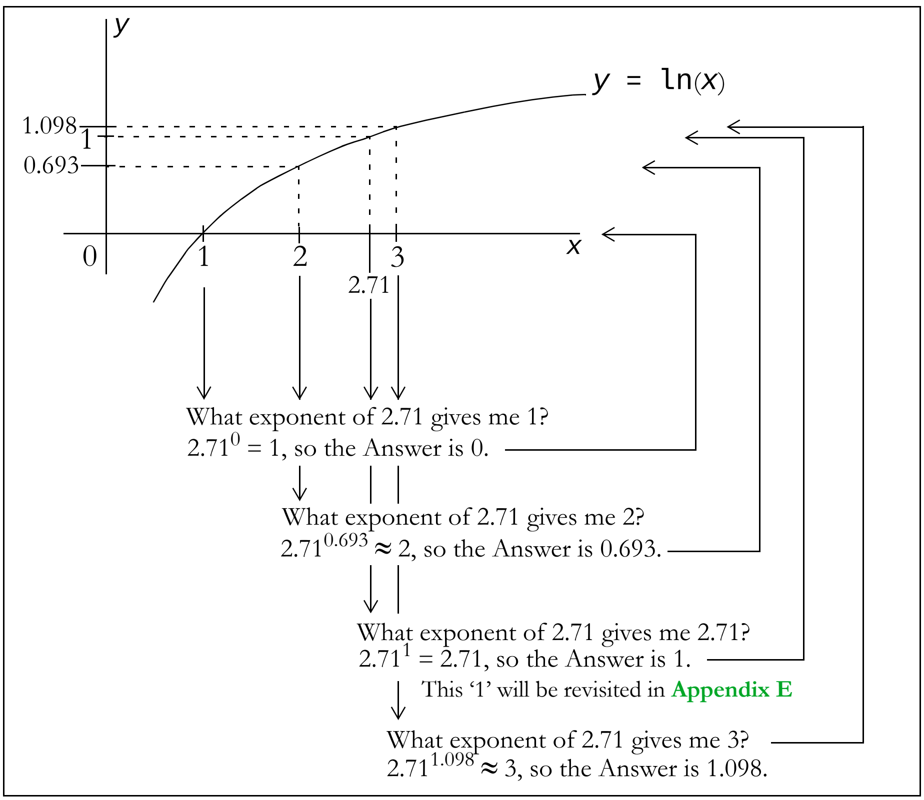
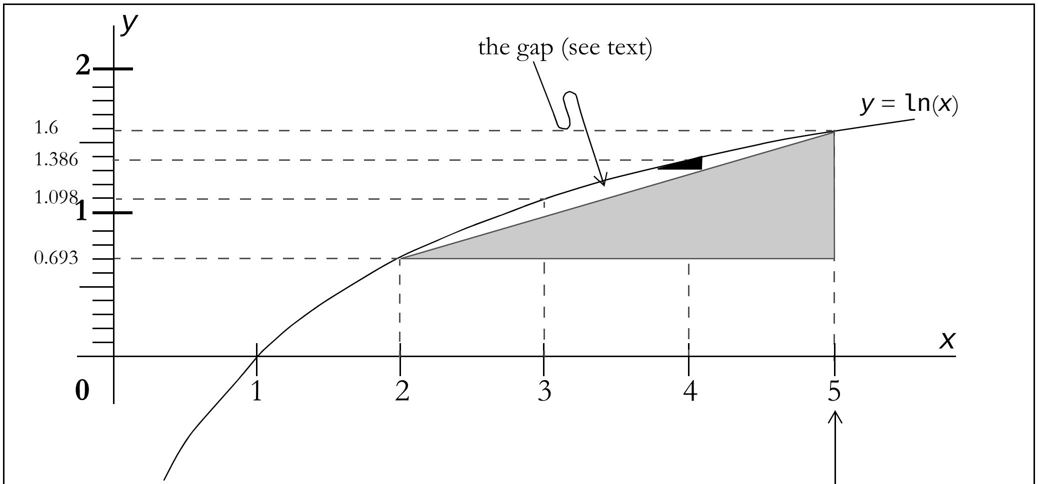


FIGURE 13: White Box View of The Natural Log Function

In Figure 13, so as not to be bamboozled by the quasi-mystical panache and gravitas of e, I pose my questions in terms of a humdrum numerical value, 2.71, i.e., the natural number *itself*. (The rationale for this quirk of mine, sometimes using 2.71 in lieu of e and sometimes using 3.14 in lieu of π I discuss in Appendices **F** and **G**.)

In **Appendix C**, Figure **72** shows another situation where one must set aside the black box notion in Figure **12** and try for a 'white box' approach instead: When working in mode radians (which is the norm in calculus), one must mentally 'supplement' the trig functions every step of the way.

The five pairs of sample values used in Figure **11** reappear in Figure **14**, associated now with functional notation instead:[9] $f(1)$, $f(2)$, $f(3)$, $f(4)$, $f(5)$. Interpretation: $f(1)$ represents the ***output*** when 1 is input to the function; $f(2)$ represents the ***output*** when 2 is input; and so on, for input values $x = 1$ through $x = 5$. E.g., you input '2' to your calculator's `ln` function, and the function's output on the calculator display is '0.693'. By all means use your calculator this way to reproduce some of the values in Figure **14**, but in doing so, try to keep in mind that there are genuine input/output scenarios (as depicted in Figure **12**) and scenarios where the input/output idea is best taken as a metaphor only, not as a closely mirrored model of how the function actually behaves.



$f(1) = \ln(1) = 0$
$f(2) = \ln(2) = 0.693$
$f(3) = \ln(3) = 1.098$
$f(4) = \ln(4) = 1.386$
$f(5) = \ln(5) = 1.6$

$f(3) - f(2) = 1.098 - 0.693 = 0.405$ (compare Figure 20 on page 40)

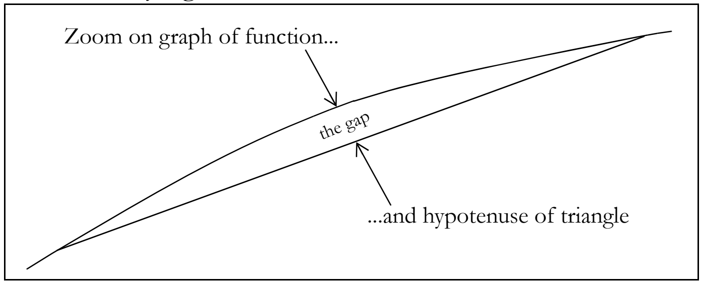
$\frac{f(5) - f(2)}{5 - 2} = \frac{1.6 - 0.693}{3} \approx \frac{0.9}{3} = 0.3 =$ slope of the large grey triangle

$f'(4) =$ slope of the small black triangle (and its value? see text)

FIGURE 14: Anatomy of a Famous Curve

In addition to the direct readouts for $f(1)$ through $f(5)$, we show $f(2)$ subtracted from $f(3)$ in anticipation of Figure 20 on page 40. Next, we relate $f(5)–f(2)/(5–2)$ to the large grey triangle. Finally, we associate $f'(4)$ with the small black triangle that is hang-gliding, as it were, at an altitude of 1.386 units on the y-axis. The slope of the large triangle is readily computed as $(1.6 – 0.693)/3 = 0.302$, or 3/10. (I.e., the triangle is a 'picture of division' as discussed in connection with Figure 7.)

However, the three decimals notwithstanding, this is a very *crude* slope, as indicated graphically by the gap in Figure 14, in-between the hypotenuse of the triangle and the curve to which it is trying to mold itself:



By contrast, the small black triangle representing $f'(4)$ (the derivative at $x = 4$) seems fairly glued to the curve, which is to say it provides a very *good* slope, since the angle of its hypotenuse is indistinguishable from the slant of the curve itself at that point. But what is the value this time? On purpose, we have shown the good slope only impressionistically for now. Conceptually, one can keep zooming in on the curve to discover the rise-over-run for smaller and smaller (better and better) triangles, but this is not always practical, and in any event there is a numeric technique that gives excellent results with far less bother: the difference quotient, alias 'the limit definition of the derivative'.

The Difference Quotient (alias 'limit definition of the derivative')

The moment of crossing from elementary mathematics into the world of calculus is marked nicely by the following passage in Hungerford's *Contemporary Precalculus*:

> [We will be using] a specialized shorthand language. Treating it as ordinary algebraic notation may lead to mistakes. — Hungerford, p. 139

The context for his warning is the difference quotient which, in its simplest form, looks like this, seemingly just a snippet of elementary algebra:

$$\frac{f(x+h) - f(x)}{h}$$

But the bland looking difference quotient, when embellished this way...

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

...can yield such a good approximation of the derivative that it becomes 'the definition of the derivative' (Stewart, p. 129) or 'the limit definition of the derivative' (as my own teacher called it). It is the variable h that tells us we are 'not in Kansas anymore',[10] i.e., no longer in the realm of *ordinary* algebraic notation, as

Hungerford warned us.

In the difference quotient, assign a very small number to h, such as 0.0001. Then the slope pops out of the calculation, in this fashion:

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \frac{\ln(4+0.0001) - \ln(4)}{0.0001} \\
&= \frac{1.386319361 - 1.386294361}{0.0001} \\
&= \frac{0.000025}{0.0001} = 0.25
\end{aligned}
$$

FIGURE 15: Difference Quotient Used To Find A Slope

No zooming required. Now we know exactly what the black triangle in Figure 14 looks like *without* ever 'going there in person' at the scale of an ant's eyebrow. The result above (0.25) tells us, from afar, that the triangle *must* look like this:

Do you detect a slight touch of magic in this procedure? I hope so.

Reality check: The overall trend of the $y = \ln(x)$ curve is to gradually level off (i.e., to become less and less steep), as it grows from left to right. Thus, if we take a rough estimate of the slope centered on the interval $x = 2$ through $x = 5$ and obtain the value 0.3, we would expect the slope near $x = 4$ to be somewhat shallower. Since 0.25 is shallower, relative to 0.3, the result of our calculation seems plausible. (Alternatively, since the derivative function for $y = \ln(x)$ is $y = 1/x$, as shown in Figure 20 on page 40, we can confirm the value for f' at $x = 4$ this way: $y = 1/x = 1/4$.)

So far we have seen the variable h used for introducing a very small value such as 0.0001 into a difference quotient calculation. The same variable, h, may also be employed to introduce the value 0 instead. The former usage pertains when calculating the slope of the graph of a function at a given *point*. The latter usage, with zero, occurs when calculating a (whole) derivative *function*, as illustrated next. Specifically, we will discover the derivative function that corresponds to (primary) function $y = x^2$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \frac{\cancel{h}(2x + h)}{\cancel{h}} = 2x + h$$

$$h = 0$$

Now it is OK to assign zero to the remaining limit variable since the other two *h*'s have cancelled one another.

$$f'(x) = 2x + 0 = \boxed{2x} \quad (\text{or } y' = 2x)$$

FIGURE 16: Difference Quotient Used To Find Derivative Function

The first line shown in Figure **16** looks identical to the case shown earlier (in Figure **15**) where we calculated a derivative (slope) at a given point. But this time we have no particular value of x in mind. Instead, we push the entire *function* $y = x^2$ through the difference quotient, using the FOIL method from elementary algebra. Rather than investigating a particular value of x we are trying to boil down the difference quotient for $y = x^2$ to its essence, which turns out to be $2x$. This in itself seems like a kind of black magic, but it gets better: Soon we will learn the arithmetic trick for going *directly* from x^2 to $2x$, without even using the difference quotient!

At that point, one might wonder if the difference quotient is only a tribal rite of passage for calculus students. No, it is more than that. It is a necessary part of one's education about the power and subtlety of limits (continued in **Chapter II**). Conceptually, h is virtually zero throughout this operation, but we must not actually *assign* zero to h until the very last step. Without taking the student through the whole difference quotient procedure there is no way to convey this important lesson about h. The following word of advice from W.M. Priestley fits the circumstance: "What has just been illustrated in this [limit] example is not hard, but it is subtle. Reread [it] to make sure you understand"; Priestley, p. 22. (Or, going just the opposite direction, I would say if this business about a derivative at a *point* versus a derivative *function* is unclear, I suggest that you should not worry about it just yet. Keep reading and the pieces should fall into place eventually.)

Recap: the 'overloaded' symbol f'

In summary, the symbol f' has come to be used with two dramatically different meanings, as determined by context:

-- the derivative at a given *point* in function f (i.e., one specific slope).

-- the derivative *function* for the entire *ensemble* of points that belong to function f. The latter is a function in its own right (which in turn has its own derivative function, thus starting the cycle anew). This 'overloading' of the symbol f' can be troublesome for the first-year student. (And so it is with the symbols $f'(x)$ and y' too since these are both synonyms for f'.)

For example, in Hughes-Hallet *et al.* we find this on one page...

Rate of change of f at $a = f'(a) = \lim_{h\to 0} [f(a+h)-f(a)]/h$

...followed by this a few pages later:

Rate of change of f at $x = f'(x) = \lim_{h\to 0} [f(x+h)-f(x)]/h$.

No one on the whole committee of authors finds it worth remarking on the *déjà vu*, even though the purpose of the former equation (on p. 74) is to define f' as the derivative at a *point* while the purpose of the latter (on p. 81) is to define f' as a derivative *function*: The very same notation has been recycled to take on a wildly different meaning.

Concerning pp. 74 and 81, specifically, in Hughes-Hallett, one might argue that the nuance of passing parameter (a) versus passing parameter (x) is enough to form the distinction I want. But the point is that generally one is much more likely to encounter a naked f' symbol, *without* its (a) or (x) qualifier. So the 'distinction' they draw is ephemeral and useless. (By the way, this overloading of the f' symbol is not an innovation of Hughes-Hallett *et al.* I cite that committee only as an arbitrary source.)

Tangent Line Equations

Slope-point form:

$$y - y_0 = m(x - x_0), \text{ solve for } y:$$
$$y = m(x - x_0) + y_0$$

where x_0 is given; y_0 is computed from the function; and m might be estimated.

Variation:

$$y = f'(x)(x - a) + f(a)$$

where a is given; $f(a)$ is computed from the function; and $f'(x)$ is computed from function, to play the role of m.

The Slope of *e*

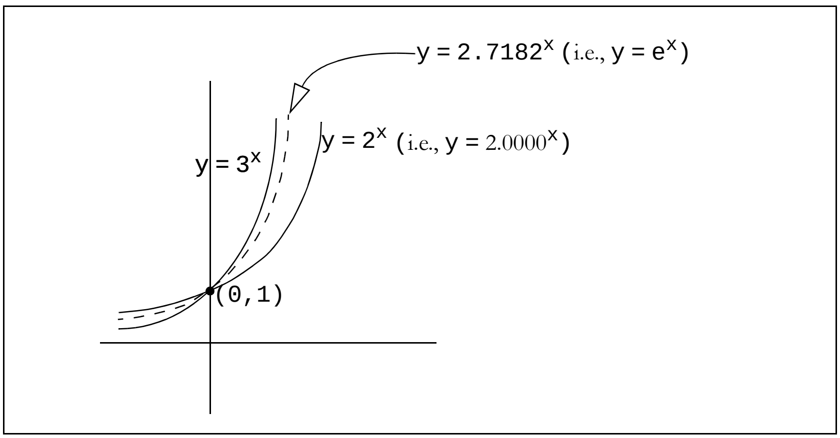


FIGURE 17: The Slope of *e* in Context (after Stewart p. 422)

When a function puts x 'in the cockpit' (as Berlinski would say, p. 80-81) it is called an exponential function. In Figure 17 we show the graphs of two exponential functions using whole numbers ($y = 2^x$ and $y = 3^x$) with a third one sandwiched in-between (dashed line as the graph of $y = 2.7182^x$). By definition, *all* exponential functions must pass through $(0, 1)$ because $1^0 = 1$, $2^0 = 1$, $3^0 = 1$, and so forth. One such case is $2.718^0 = 1$, where it happens that the slope, *too*, is 1. Ultimately, this is what lies behind a curious looking statement found in all calculus texts to the effect that 'the function $f(x) = e^x$ is its own derivative' (e.g., in Stewart, p. 422). For more about this, see **Appendix E**.

II Limits

This short chapter contains only the bare essentials on limits. Some of their more engrossing aspects are covered elsewhere, in the discussion of the difference quotient (page **27**f.); in the **Dead Leaf Density** problem in **Appendix A**; and especially in **Appendix D: Imposed Limits, Inherent Limits**. Limits also play an important role in the definition of 'integral' on page **218** and in the presentation of Green's Theorem that begins on page **101**.

The Little-δ Little-ε Picture

Consider the function $y = (1 + h)^{1/h}$. Using limit notation as introduced in **Chapter I**, one may embellish the function as follows...

$$\lim_{h\to 0} (1 + h)^{1/h} = ?$$

...this being an invitation to discover the value of the function at its limit as h approaches zero. One way to proceed would be to build a table of input/output pairs, to see how the function behaves using values of h that are progressively closer and closer to zero:

h	y
.1	2.59
.01	2.70
.001	2.71
.0001	2.7181
.00001	2.7182
.000001	2.71828
.0000001	2.71828

Notice how the value of y seems to get 'stuck' toward the bottom of the pyramid, with the decimal pattern '71828' repeating itself. This makes us suspect that no matter how closely h approaches zero, y will never again vary by more than some miniscule amount from 2.71828. And if one happened to recognize ‘71828’ as the first five decimals of e, that would clinch one’s surmise. It seems reasonable to assume that the limit in this case is the natural number, e.

Alternatively, a limit problem might be posed in this fashion:

As x approaches ∞, evaluate $f(x) = (x + 3)/(2 - x)$

.

Here one might try introducing two or three large numbers such as 1000 and 1000000 as modest approximations of ‘infinity’, and see how the function behaves:

$$f(1000) = 1000 + 3 / 2 - 1000 = 1003 / -998 = -1.005$$

$$f(1000000) = 1000000 + 3 / 2 - 1000000 = 1000003 / -999998 = -1.000005$$

Already with just these two trial values one has a strong impression that the function must evaluate to –1. Where else could such numbers possibly lead?

In some cases one can safely surmise a function’s value at the stated limit just by eyeballing it. E.g., $f(x) = 3e^x + 2 / 2e^x + 3$ for $\lim x \to \infty$ looks as though it should evaluate to 3/2, and it does. In other cases, algebraic manipulation might be required to confirm one’s surmise (or to methodically work one’s way around to a plausible answer in the first place). E.g., it is not immediately obvious (to me at least) that $\lim x \to \infty\ f(x) = (x^2 + 2x - 1)/(3 + 3x^2)$ evaluates to 1/3. But algebra confirms this.

In generic form, a limit evaluation may be represented as follows:

$$\lim_{x \to c} f(x) = L$$

Aside: *If* represented by L, as defined by the equation immediately above, our answers such as –1 and 1/3 above would honor the limit *as* a limit. Note, however, that our problems involved a subtle shift in emphasis, as they asked the student to ‘evaluate the function’, *not* ‘find its limit’. This is what I call the ‘flea-hop’. As discussed in **Appendix D**, I object to this practice of flea-hopping *onto* the limit and proclaiming that the function *itself possesses* such-and-such value. (I’ve allowed myself to illustrate it above only because it is one of the ‘normal’ ways of discussing limits that a student must learn, whatever my own opinion of it may be.)

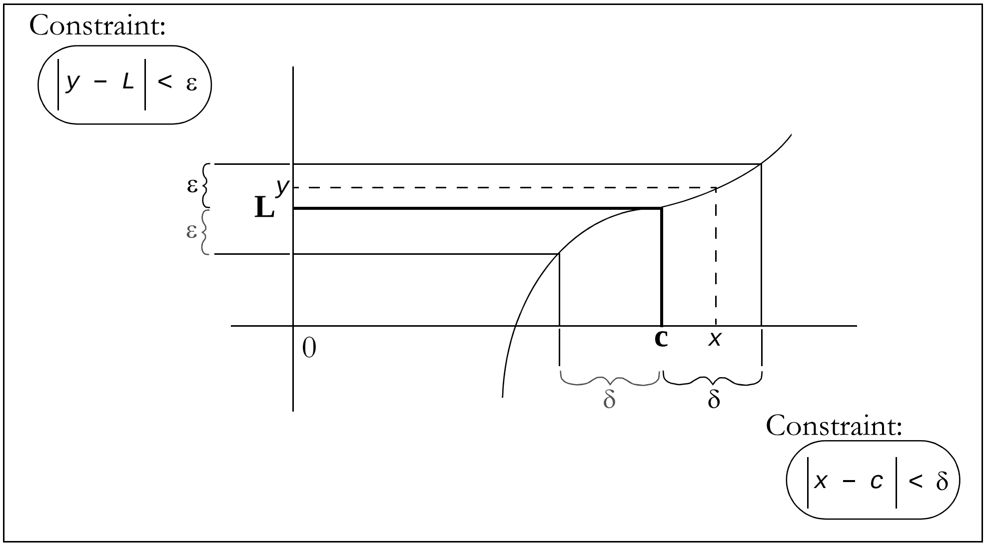


FIGURE 18: A Picture of the Limit at Close Range

In Figure 18, we've shown the case where x lies to the right of c (but somewhere within the zone labeled δ),[11] which dictates that y lies above L (but somewhere within the zone labeled ε). Implied but not illustrated is the case where x lies to the left c, which dictates that y lies beneath L (in which case the other zones labeled δ and ε would come into play, whereas in this diagram they only go along for the ride, so I've greyed them out).

Stated more formally: The absolute value of y minus L must be less than *epsilon* and the absolute value of x minus c must be less than *delta*. (This translates the two sets of symbols enclosed by oblong boxes in Figure 18.)

Properties of Limits

Limits can be added, multiplied, or divided:

$\lim_{x\to c} (bf(x)) = b\left(\lim_{x\to c} f(x)\right)$	In the vernacular: "It's okay to pull b through the lim symbol."
$\lim_{x\to c} (f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$	Addition
$\lim_{x\to c} (f(x)g(x)) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right)$	Multiplication
$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ provided $\lim_{x\to c} g(x) \neq 0$	Division (but not by zero)
For any constant k, $\lim_{x\to c} k = k$	Example: $\lim_{x\to 3} 9 = 9$
$\lim_{x\to c} x = c$	Example: $\lim_{x\to 3} x = 3$

Limits, Continuity, and Differentiability

From algebra classes we have the acronym **LCD** meaning Least Common Denominator. In calculus, it helps if we recycle that acronym, now letting it stand for the following...

Limits, Continuity, Differentiability

...since these three subtopics are likely to appear together on a quiz or exam paper, early on in Calculus I. Here is the hierarchy of rules that ties the three subtopics together:

- *Limits*: no jumps or gaps, but holes are OK (one may 'pick up the pencil' to jump over 'a pinhole in the paper').
- *Continuity*: no jumps or gaps or holes, but corner or pinch (cusp) or vertical line (nonasymptotic) is OK.
- *Differentiability*: no jumps or gaps or holes or corners or pinches or vertical lines

The rules thus far are summarized in Figure 19.

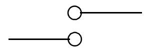
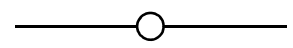
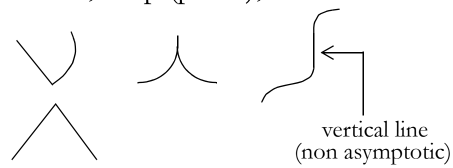
Limits:	no gap  No jumps or gaps, but holes are OK
Continuity:	no hole  Corner, cusp (pinch), or v-line OK 
Differentiability:	no corner or cusp allowed if function is to be declared differentiable 

FIGURE 19: Limits, Continuity, Differentiability

For an example of a cusp, see Figure **32**, top of column F, and the ensuing discussion on page **56**.

For the '*Continuity*' rule, there are three crucial sub-criteria to know about...

- **D***efined*: The function must be defined at $x = c$.
- **E***xist*: $\lim_{x \to c} f(x)$ must exist.
- **E***quivalent*: The limit and value must be equivalent: $\lim_{x \to c} f(x) == f(c)$.

...for which one might use the acronym **DEE** (not to be confused with **DNE** = Does Not Exist, in reference to limit tests).

(Regarding the double equals sign (= =), see page **194**.)

Finally, there are two sub-sub-criteria attached to the '*Exist*' rule above:

- Lefthand *and* righthand limits exist.
- Lefthand and righthand limits are *equal*.

(Source for the first three bullets: any calculus textbook; source for the latter five bullets: Robinson *et al.*, p. 79-81.)

That's all we'll be saying about limits from the Calculus I perspective. For the Calculus II perspective and others (cosmic? *Killer Klowns from Outer Space*?), see **Appendix D**.That's where the real discussion begins.

III The Fundamental Theorem of Calculus (FTC)

A Pictorial Approach (Mainly) to the FTC

At first glance, Figure **20** may seem to involve two pairs of curves but really it is the same pair repeated. In Figure **20**a, the 'You Are Here' sign is associated with the rising curve. In Figure **20**b, the 'You Are Here' sign is associated the falling curve.

The other difference is in the labeling of the curves themselves: labels f and f' in Figure **20**a are replaced by labels F and f, respectively, in Figure **20**b. Otherwise, nothing changes. We are just presenting two different perspectives on the same material, as detailed below.

The rising curve is the graph of the function $y = \ln(x)$. It should look familiar from Figures **11** and **13** in **Chapter I**. The falling curve is new. In Figure **20**a we see the falling curve in the role of derivative function (f') of the ln function. In Figure **20**b we see it as a function[12] in its own right, $f = 1/x$. (This bland-looking function turns out to possess an illustrious history; see **Appendix E**.)

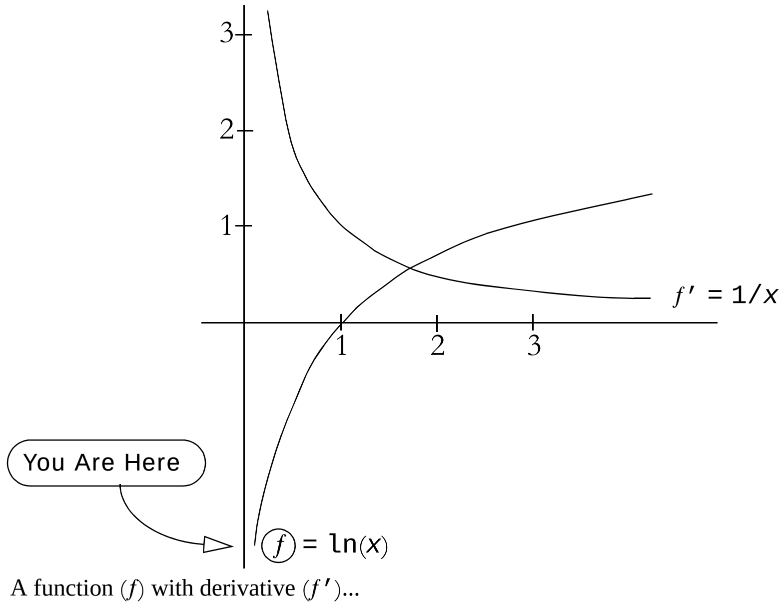


Figure 20a

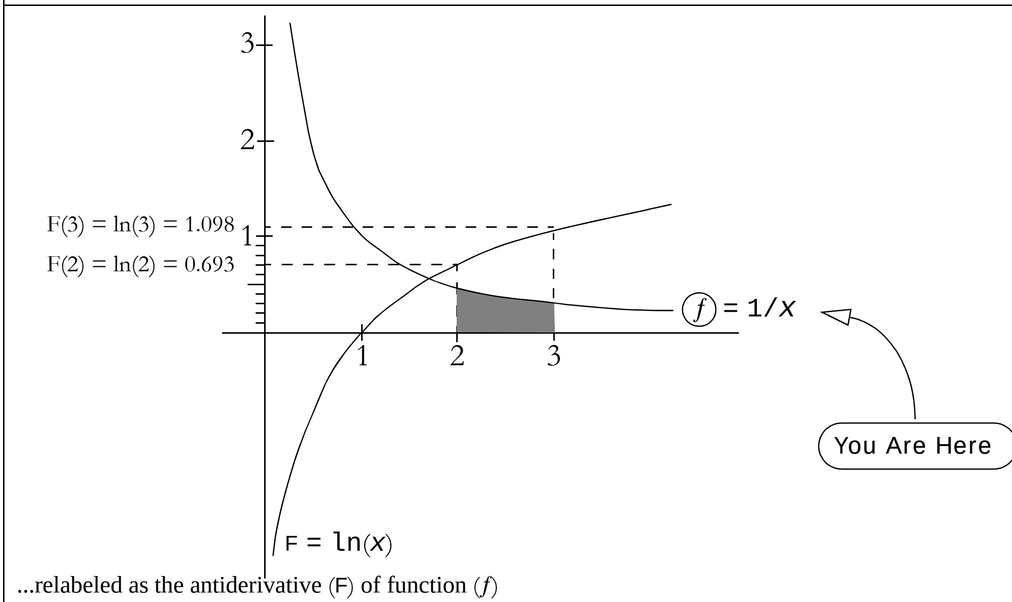


Figure 20b

FIGURE 20: The FTC Illustrated by $f = \ln(x)$

In Figure 20a, we are dealing with the same kind of relation described in connection with **Figure 16 (Difference Quotient Used To Find Derivative Function)**. However, in this case we have omitted the specifics; you'll need to take the relation on faith.

When we move down to Figure **20**b, we travel from the world of function/derivative pairs (labeled f/f') to a new world: that of antiderivative/derivative pairs (labeled F/ f instead). It is in the arena of Figure **20**b that the FTC may be illustrated, as follows: Suppose one wishes to determine an area under the curve of $f = 1/x$, say from x = 2 to x = 3. By switching to the sister function, F = ln(x) and providing the values 2 and 3 as input to *it* (counter intuitively), we can obtain the desired information about $f = 1/x$:

```
F(3) = ln(3) = 1.098
F(2) = ln(2) = 0.693 –
               0.405
```

That's a quick and easy way of computing the area of the shaded area under the $f = 1/x$ curve. Without the FTC, we could eyeball the area, noting that it must be something less than 1/2 of a 1 x 1 square, as defined by 2 and 3 on the x-axis and by 0 and 1 on the y-axis. But beyond that we would be at a loss to be more specific. The idea is to find the value of the antiderivative F = ln(x) at $x = 3$ and at $x = 2$, then subtract the two numbers. Each line involves just a few key-presses on a calculator, as discussed (with caveats!) in **Chapter I** already. The method works because of the relation guaranteed by the FTC between *any* such F and f pair.

Now, *for an engineer*, the excitement of the FTC is that it allows one to perform an end run such as the one described above. Meanwhile, *in nature* (for lack of a better contrastive phrase), the relationship must be something quite different, as suggested by Figure **21**.

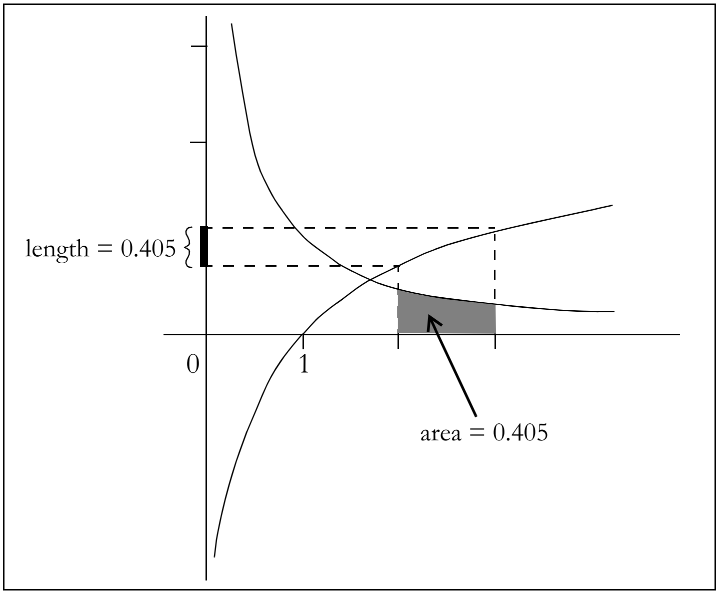


FIGURE 21: Length 0.405 Corresponds to Area 0.405

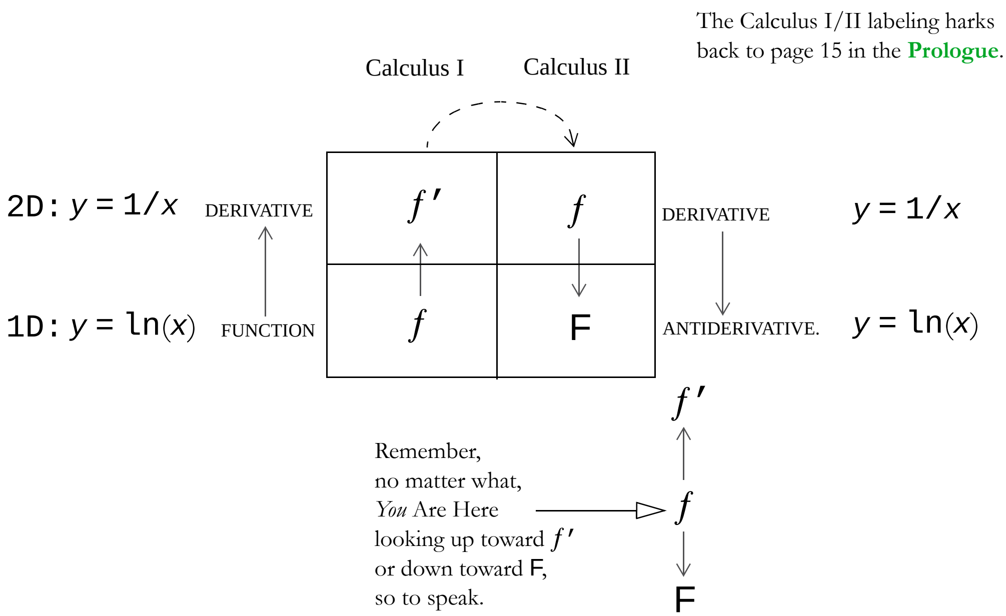
Nature does not 'need' or 'care about' about our 1.098 minus 0.693; that's a human being's extra, clunky step. In nature (metaphorically speaking) we're already at 0.405 from the git-go. The value 0.405 is not 'an answer'; it is simply a *length*, a piece of a line. The message of Figure **21** is one of those things that is 'so obvious it is easily missed', namely:

length 0.405 corresponds to area 0.405

Or, as I would prefer to say, *area is explained by length* or better yet, *2D is explained by 1D*. The latter is best because it can easily be shifted to other dimensional situations — and there are many such to consider as we'll see in **Chapter VII**. But although we'll encounter some fairly mind-boggling variations on the theme later, Figure **21** is essentially 'it' — what this whole book is about. If you've taken to heart all your science instructors' admonitions about the importance of dimensions and units, the relationship it shows will have to impress you as something just short of voodoo.

The Fourfold F Revisited

If we review Figure **6** in the context of Figure **20**, it should make more sense now. For the reader's convenience, I've reproduced the left side of Figure **6** as Figure **22**, next.



The curved dashed line joining f' to f serves two purposes: First, it suggests a little 'story' of how the whole set of four symbols arises, shown here as a straightened-out, left-to-right progression with a semblance of order:

$f \longrightarrow f'$ $f \longrightarrow \mathsf{F}$

Secondly, the curved arrow is a reminder that the derivative gets ***gratuitously*** renamed as we move from the column labeled 'Calculus I' to the column labeled 'Calculus II', thus causing f to be an overloaded symbol, just like f' (although the latter is overloaded in an entirely different way, as described on `page 29`). Think about it: That leaves only F as a pure symbol that doesn't rely on context to scare up a meaning!
Editorial comment: Is it any wonder that the student of calculus is sometimes confused? This scheme they are expected to internalize is a notational travesty. And, at this stage, we haven't even mentioned the whole quirky business about FTC Canonical Form versus FTC Pragmatic Form (page 210) which comes up in Calculus III, in sphinx-like silence, just when one thinks she might be beginning to understand the lay of the land.

FIGURE 22: The Fourfold F Again, Now Aligned with 1D and 2D

In first-year calculus, it is quite natural for the student to develop the notion that the derivative function f belongs always to a higher dimension than its antiderivative F, as occurs in Figures 20-22, for example, where a problem that originates with 2D derivative function $f = 1/x$ is 'explained by' or 'solved by' its 1D antiderivative $\mathsf{F} = \ln(x)$. But in Calculus III the expected relation (for n dimensions relative to

n–1 dimensions) gets turned around frequently. In problems where Green's Theorem is applicable, it happens that the lower-dimensional side of the equation where we see a single integral sign ∫ is generally *more* difficult to evaluate than the higher-dimensional side where we see ∬, a double integral.*

So, strange or not, one quickly learns to use Green's Theorem to evaluate the *latter*, of course. Thus, using the terminology introduced above, I would have to say something like '1D is explained by 2D' (for details, see Table **7** and Figure **62**). That does not sound interesting or surprising the way '2D is explained by 1D' does, but there are plenty of other aspects of Green's Theorem that will more than compensate for the seeming anticlimax; see pp. **101-117**.

Returning to the main thread of discussion, here is the Fundamental Theorem of Elementary Calculus (the FTC), flanked by the two halves of our concrete example:

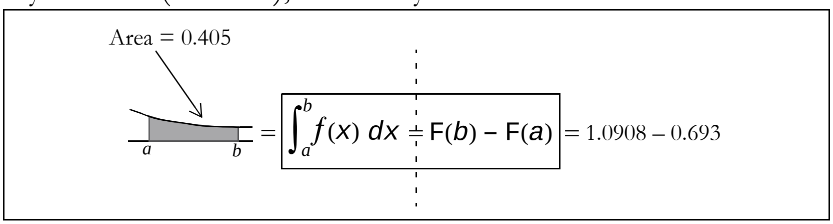
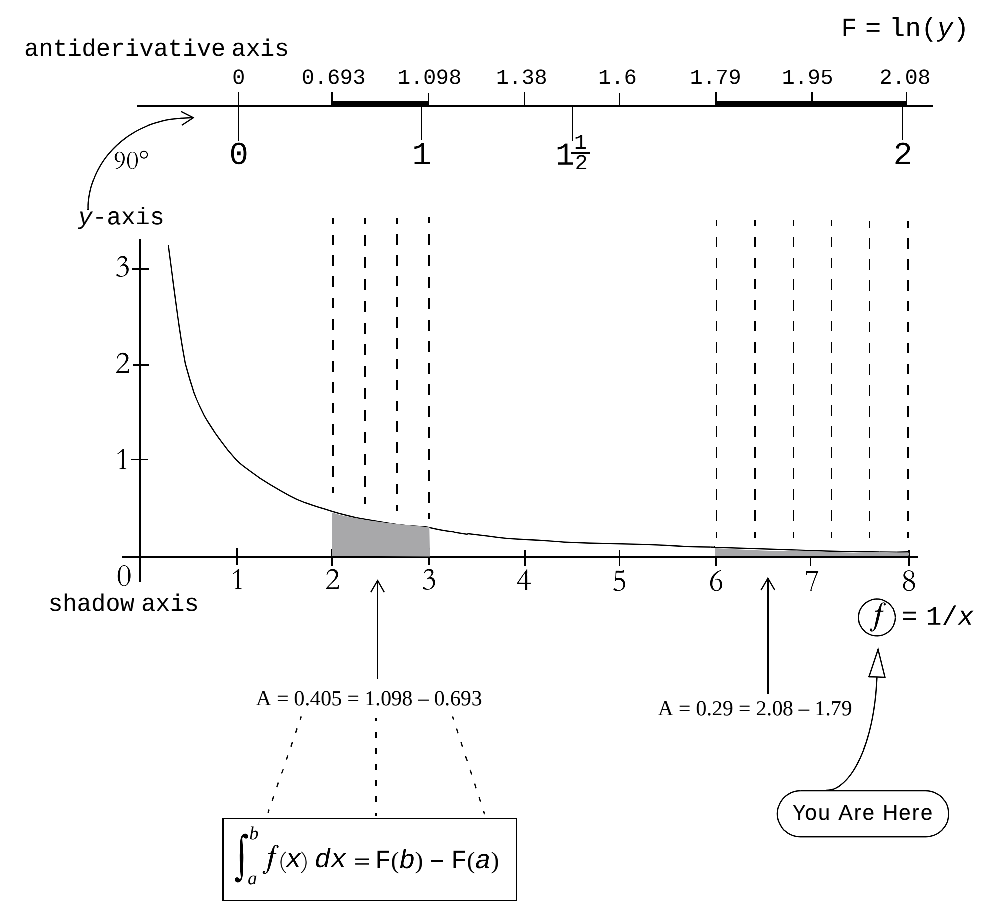


FIGURE 23: The Fundamental Theorem of Calculus (FTC)

In Figure **23**, the left side of the theorem, $\int f(x)\,dx$, corresponds to the shaded area under the curve. The right side of the theorem, $F(b) - F(a)$, corresponds to 1.098 – 0.693 or to 2.08 – 1.79 in Figure **24** where I show two such calculations together. Except for the second shaded area at $x = 6$ to $x = 8$, Figure **24** simply repeats information from earlier ones in a new format. Here we are trying to represent the human being's viewpoint (Figure **20**) and nature's presumed viewpoint (Figure **21**) together in a single picture. To do so, I've moved the y-axis data around to an 'antiderivative axis' that whimsically 'casts its shadow' on the x-axis. The old x-axis is therefore now relabeled as the 'shadow axis'. This is to dramatize the intimate yet counterintuitive relation between the two (as presaged by Figure **1**).

* The symbol ∫ is the integral sign, to be described in **Chapter V: Integral Calculus**. See alsoFigure 43 in **Chapter VI: Rules** and **Appendix F**.



$f = 1/x$ with addition of its antiderivative as a 'shadow-casting axis' up above
(≈ Figure 20 with y-axis repeated, twisted 90°, and rescaled)

FIGURE 24: An Antiderivative (aka the Original Function) Casting Its 'Shadow'

Once you've seen the FTC *this* way, you have the necessary foundation to develop a ***visceral*** understanding of the various Calculus III topics that are summarized in Figure **63** on page **127**. For instance, questions about a surface integral that curves through space can be answered by examining the 'shadow' it projects on a 2D area 'down below'. (Compare Figure **33** on page **56**.)

In Figure **24**, we have a real example that is closely analogous to the original diagram in Figure **1**: One's task is to determine the shadow's area (an onerous task) but rather magically it is the palm tree that provides the value you seek (by a simple measurement of its height with a string, let's say).[13] From the vantage point of the present chapter, one can see that the Figure **1** metaphor is surprisingly apt as such things go, even if it is whimsical. Equating height (or length) to area is exactly what the FTC does, among other remarkable things. (Compare page **136** in **Chapter VII**, where the concept is transported to a higher dimension.)

Given that the focus in this book tends toward the visual, the qualitative, and the nonnumeric aspects of calculus, the following caveat is especially important in our context: 'The integral of f is not always an area. The fundamental theorem asserts that the antiderivative method works even when the function f is not always positive' (Priestley, pp. 266-267). And here Priestley offers yet another way to think about the FTC: 'The fundamental theorem shows the connection between the two branches of calculus, ***differential*** and ***integral***' (p. 265).

A final remark about FTC notation. Perhaps it seemed churlish when I indicated, in Figure **22**, that the derivative gets '*gratuitously*' renamed in the conventional scheme, but consider this: In Protter and Morrey, p. 445, we see the FTC notated (without comment or apology) as follows:

$$\int_a^b f'(x)dx = f(b) - f(a)$$

In other words, the *two* symbols f' and f are quite adequate all by themselves for doing the job of expressing differentiation and integration. Moreover, this economical and prime-preserving style of notation is superior because what *matters* is that we recognize the thing under the integral sign as a *derivative* function. What matters hardly at all is that the thing on the right happens to be a (so-called) 'antiderivative', the latter being an almost meaningless term (see page **203**). If you have time for nothing else in this book, at least take a good long look at the formula above, for it speaks the truth. In the ensuing pages, I will return reluctantly to the conventional notation which is at best pointless, at worst criminally neurotic or pathological.

IV Curves!

The Curvature Kartouche — first look

Many of the curves of calculus can be related to one another through a device I call the Curvature Kartouche,[14] which is depicted in Figure **25** (and on the back cover of this book). Its Legend is provided separately in Figure **26**. The chapter concludes with four examples showing how the kartouche and 'slide rule' can help one answer a certain kind of exam question that comes up in Calculus I. At this point, though, we're still just doing a conceptual overview of the device.

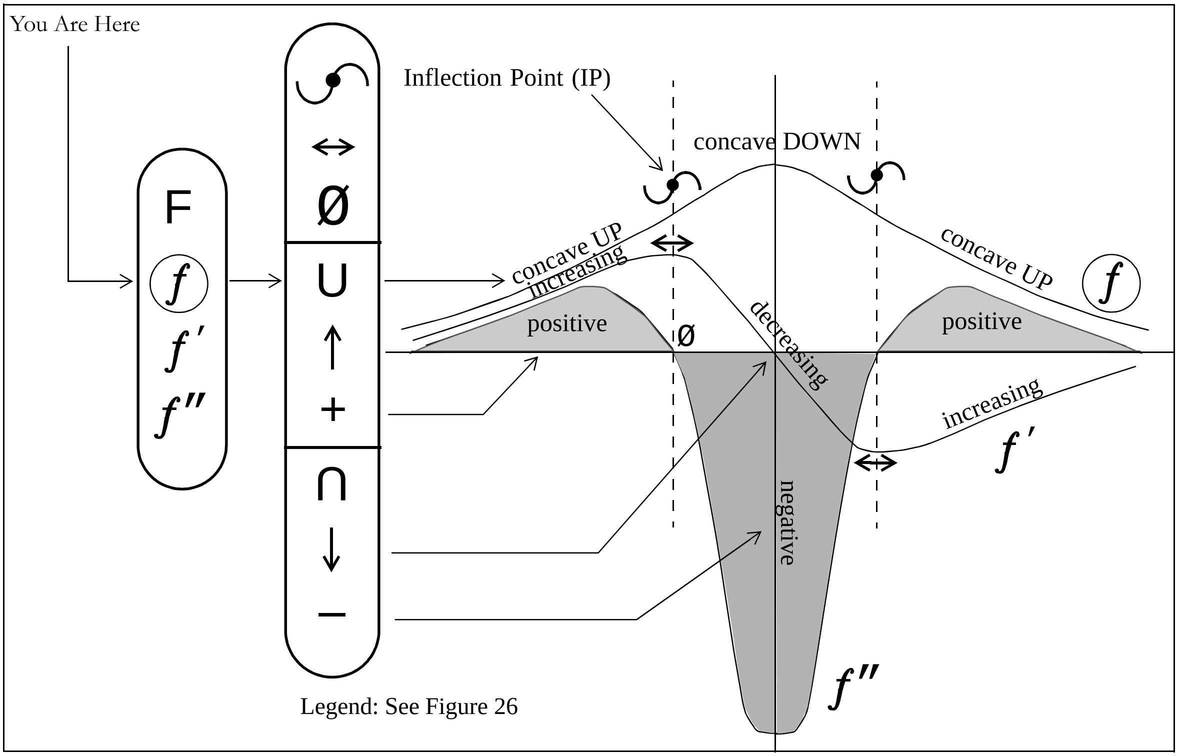


FIGURE 25: Kartouche with 'Slide Rule' to its Left and Illustrative Curves to its Right

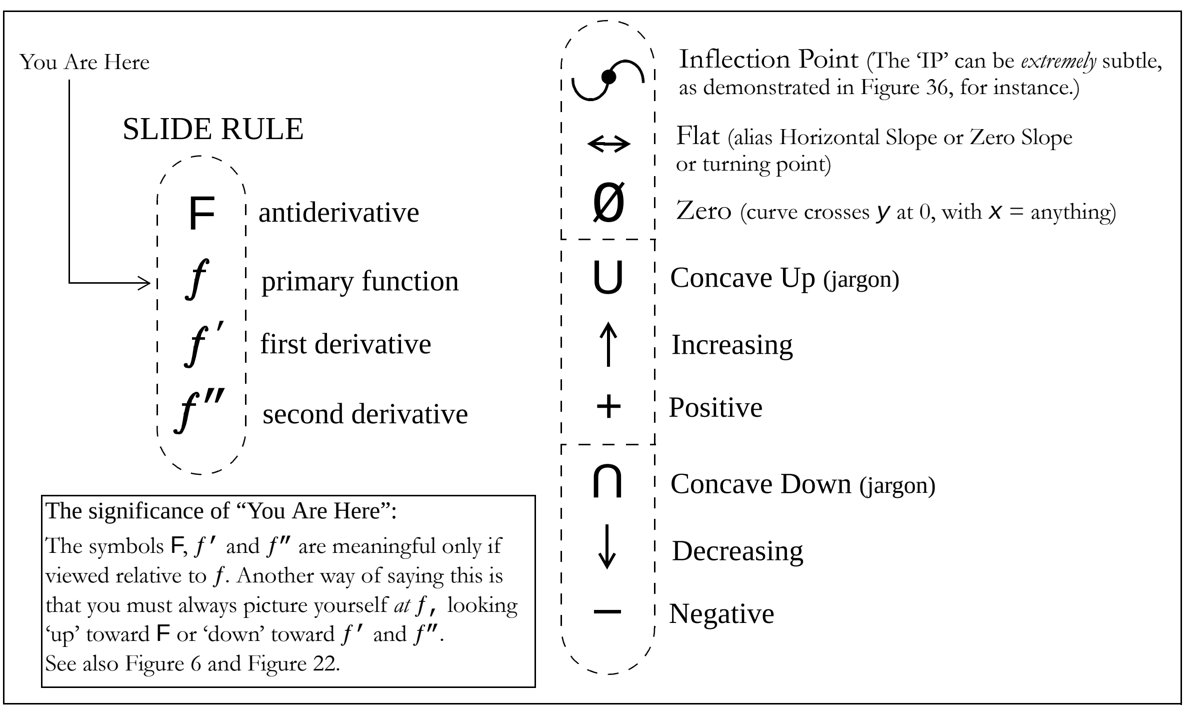


FIGURE 26: Legend for the Kartouche and 'Slide Rule' in Figure 25

At first glance, the 'slide rule' and 'kartouche' in Figure 25 might seem overly terse, even hopelessly cryptic, but this is a Good Thing: Into this one graphical device I've distilled a whole collection of calculus rules that you might find scattered over a half dozen pages in the typical textbook. Invest a few minutes of time to learn the Legend in Figure 26, and you have a practical tool. In short, it is a crib — a crib on steroids if you like. To use the tool, realize that the 'slide rule' portion (on the left) is to be notionally shifted (or physically shifted, if photocopied and scissored out) into many different positions, as suggested by a few random examples in Figure 27, illustrating its use with two or three curves at a time. Constraint: The two or three icons that you aim at must reside within one of the three original 'trios' of the kartouche itself, as indicated by the dashed horizontal boundary lines in Figure 27: Those two imaginary borders may be crossed at will but never straddled, since straddling would result in a nonsensical 'answer' to an exam question.

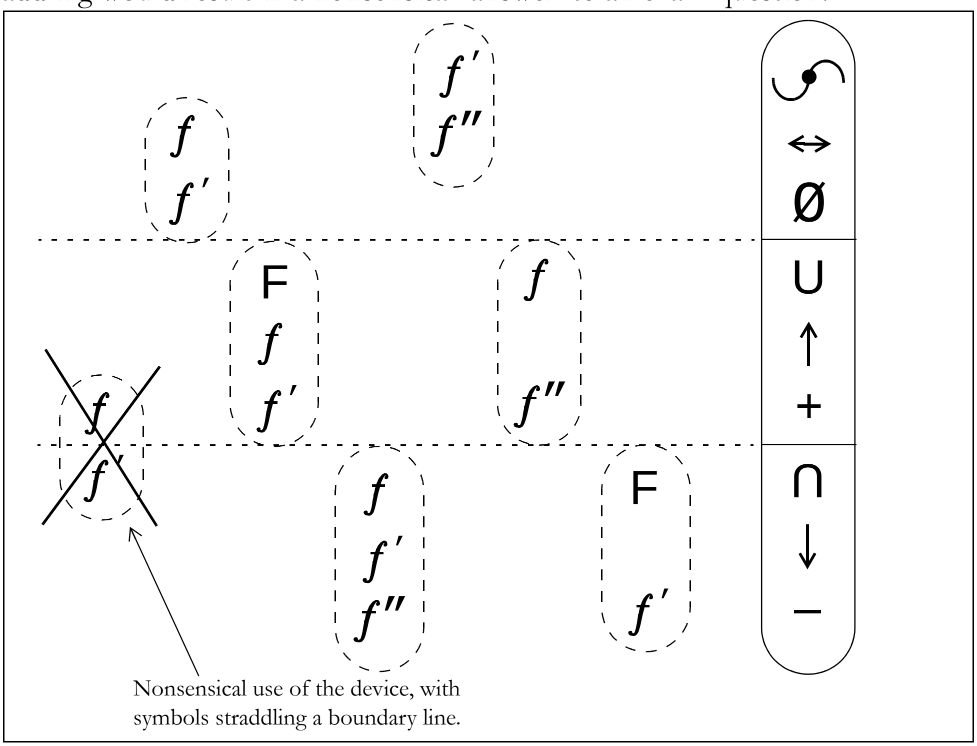


FIGURE 27: The Slide Rule in Action (Six Random Examples)

Typically the positions of interest will be contiguous, but exceptions are certainly possible, as indicated by the two rightmost examples in Figure 27 which have deliberate gaps in them. Complete examples of how to use this device are provided later in the chapter, as Figures 34 through 37.

In Figure 28 I show a variation on the kartouche idea that is worth considering. This is a more utilitarian version; in fact, this is the one I used as a crib when I was a student, before distilling it to the format used in Figure 25 above (which I think is less cryptic and more appealing to the eye).

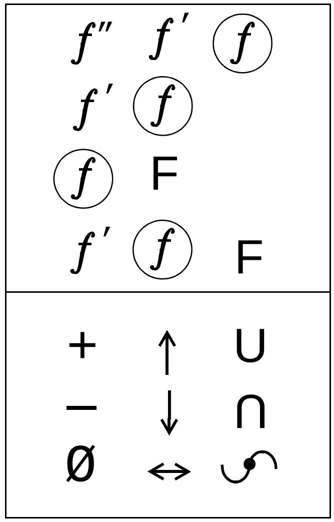


FIGURE 28: Static Kartouche (A version that involves no slide rule)

In Figure 28, there are no moving parts, so to speak. In lieu of using a slide rule, the idea is to mix and match one row from the upper portion of the (static) table with one row from the lower portion, in your mind, as needed to answer questions about a given function or curve.

Why the circled symbols? Around each instance of f I've placed a circle as a reminder that f always means 'You Are Here' (as discussed earlier). The gaps in rows 2 and 3 are on purpose. Generally, the rows are to be recited from *right to left* (e.g., $f\ f'\ f''$ is matched to Up, Increasing, Positive and to Down, Decreasing, Negative and to IP, Flat, Zero). Note in passing that *if* the symbols in Figure 28 are read vertically, they reveal an important pattern that is necessarily obscured in Figure 25: Positive–Negative–Zero parallel to Increasing–Decreasing–Flat (alias Turning Point) parallel to Up–Down–Inflection Point. Thus, the kartouche in Figure 25 and the 3 x 7 array depicted in Figure 28 each has its strengths and weaknesses for one to weigh in choosing between them.

Curves 101

As indicated in Figure 29, there are four seemingly simple curves that take on subtly different meanings depending upon whether one's perspective is that of First Derivative Geometry or Second Derivative Geometry.

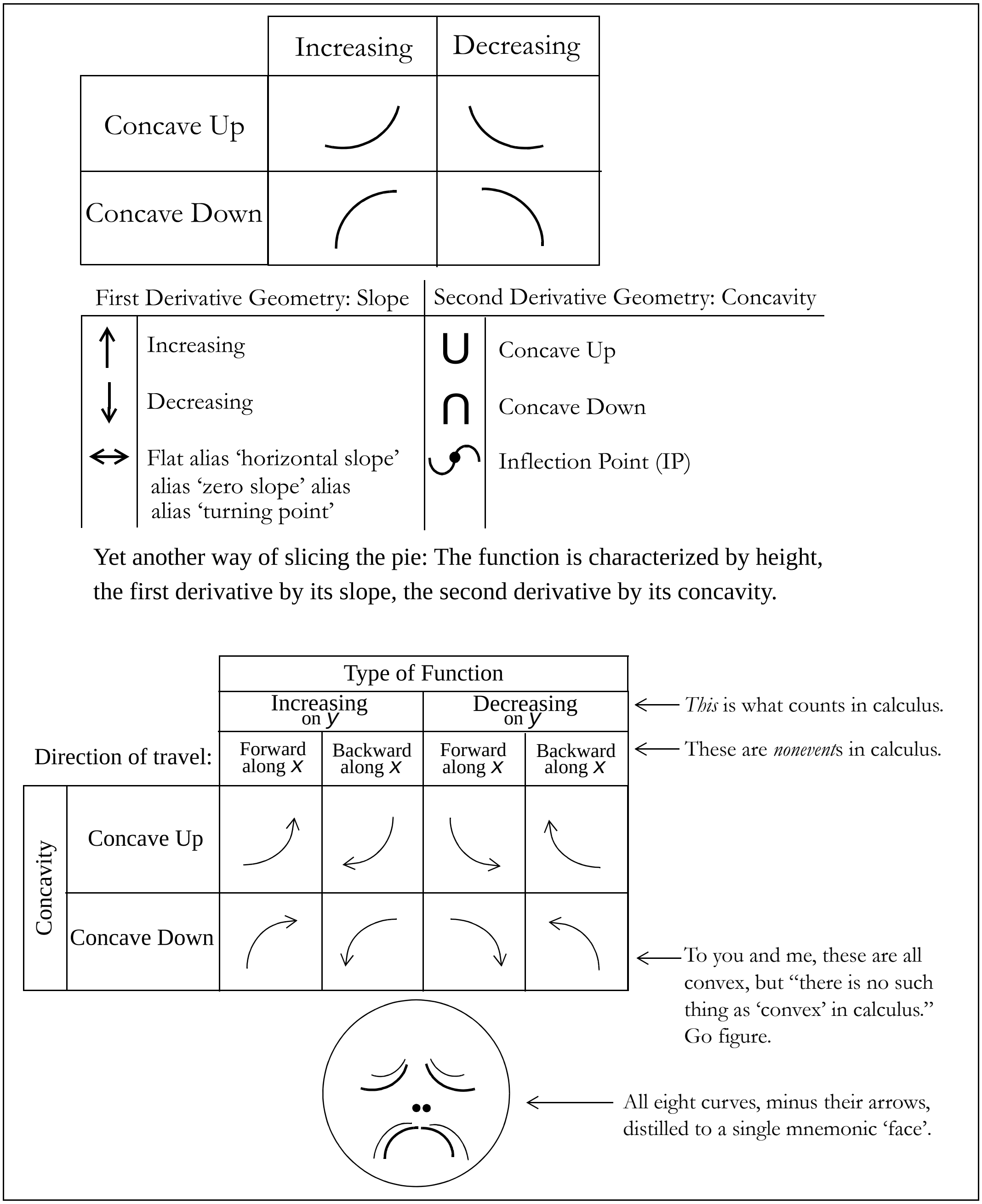


FIGURE 29: Curves 101

For additional perspective on the nomenclature used in Figure **29**, see **Appendix G: Glossary of Jargon from Antiderivative to Wonk**.

Lining Up the Usual Suspects Three by Three

The relationship of the grouped calculus curves in Figure **31** bears a certain resemblance to the relation between certain pronouns in English, as summarized in Figure **30**.

		(A)	(B)	(C)	(D)	(E)	(F)
(1)	SUBJECT	I	she	we	you	they	it
(2)	OBJECT	me	her	us	you	them	it
(3)	POSSESSIVE	mine	hers	ours	yours	theirs	its

FIGURE 30: Trios of Pronouns

Here we have six 'trios' of pronouns, one trio per column. With repeated exposure, the native speaker comes to *feel* that 'they/them/theirs' or 'we/us/ours' constitute a reasonable trio, but one would be hard-pressed to come up with a single, concise, cold *logic* rule that explains to the foreign student of English how to fill in rows 2 and 3, given row 1. (Row 3 hints at an –*r*'s rule, but is muddied by several exceptions. Most of row 2 seems to have no rule at all relative to row 1!)

Similarly, with repeated exposure one may come to feel that the trios of calculus curves presented in Figures **31** and **32** form 'natural' or 'reasonable' shape-families even though there is no easily articulated rule for getting from a shape in row 1 to a shape in row 2, or from row 2 to row 3. To the contrary, trios A, B and E go from complex to simple, while the trend in trios C, D and F is, if anything, the opposite. This may seem slightly illogical at first glance, but then so is the (non)pattern of our English pronouns in Figure **30** which are also 'part of nature' (the nature of the human animal chattering).

(What happened to the palm-shadow analogy introduced in the **Prologue**? That analogy permitted only two members per 'family'. To acknowledge the presence of the second derivative, we now need more than two members per family, so I've switched to a richer analogy.)

Picking up where we left off on page **5**, let's call Figure **31** 'a page from the Book of

Nature'. Having discovered calculus, humans place curves like those in Figures **31** and **32** in their textbooks. But really these are just part of the world at large, a deep-seated pattern *in nature*. Their technical aspect is covered elsewhere in this volume, e.g., via Rules for differentiation and integration (**Chapter VI**); this chapter is devoted to the enjoyment of their beauty.[15]

In Figures **31** and **32**, each trio (labeled A, B, C...) is a set of intimately related curves.[16] I show the members of each trio first in isolation (1, 2, 3) then together as a family with labels $f\ f'\ f''$.

According to Peter Stucki, a colleague of mine at Medtronic, 'Unhappy families are all alike; every happy family is happy in its own way.' In other words, he would reverse the famous aphorism from page one of the novel *Anna Karenina*. His notion is based on modern research, as exemplified perhaps by the families portrayed in *Six Feet Under* and *The Royal Tenenbaums* and *Little Miss Sunshine*?). Be that as it may, here are some happy families:

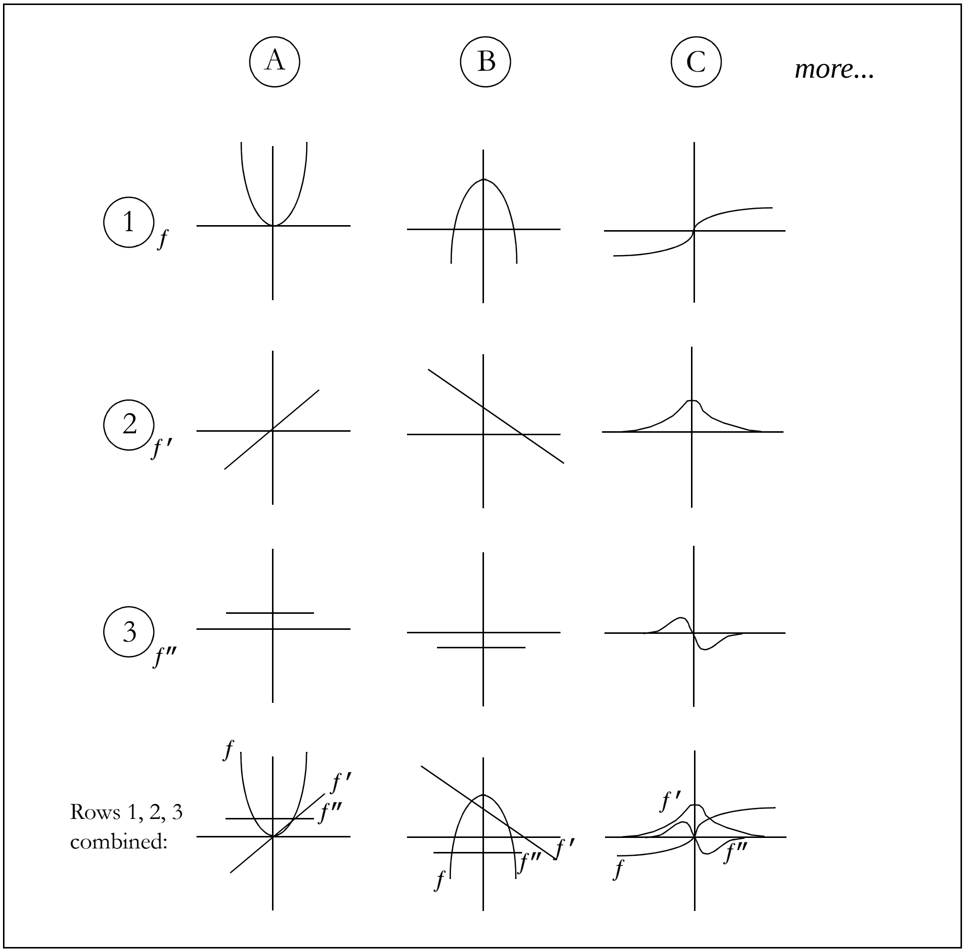


FIGURE 31: Some Functional Families

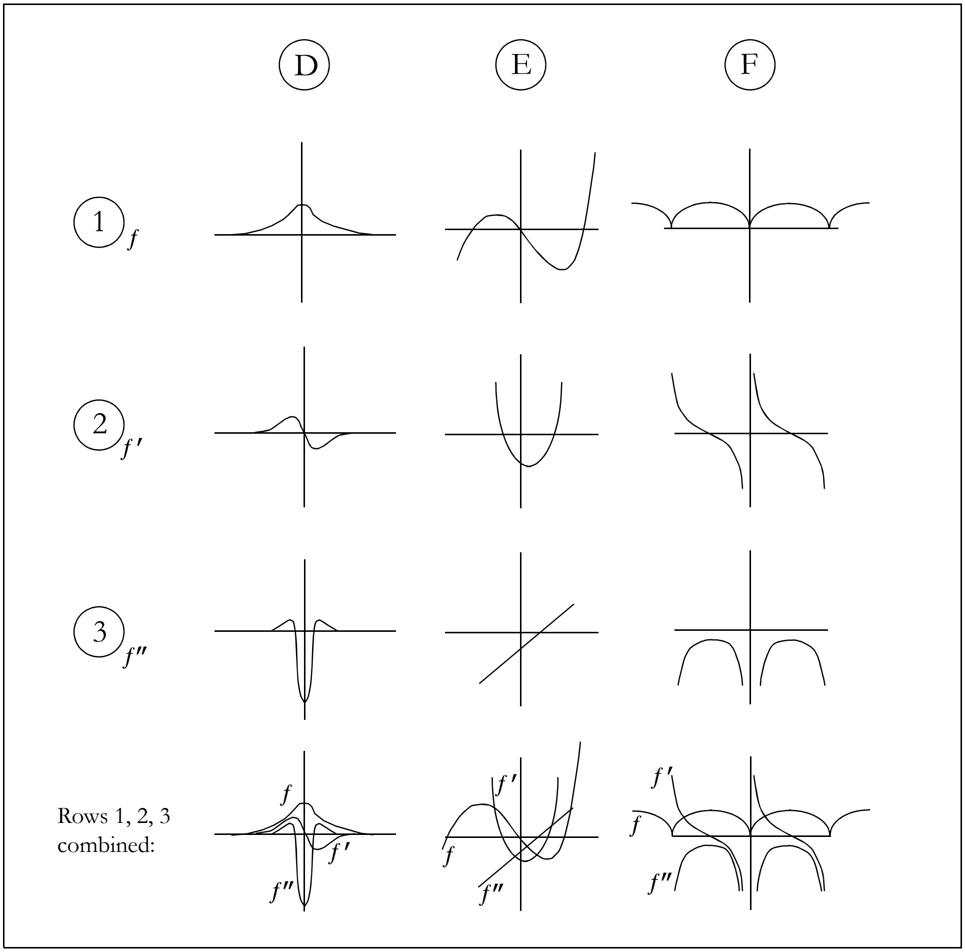


FIGURE 32: Some More Happy Families

Why three per 'family'? Because three is enough in many situations of interest to humans (e.g., in economics, chemistry or physics), although in theory there is no particular reason to stop at three; the number of curves in such a family is unlimited.[17]

Classroom Activity idea: Make photocopies of Figures 31 and 32. Scissor out each curve as a kind of 'scrabble' piece. Mix up the calculus curve scrabble pieces and deal them out to students. The object is to 'spell' one complete trio of curves as a 'word' to be verified later against the composite images in row 4 of the two figures.

Most of the graphs in row 1 of Figures 31 and 32 are produced by garden-variety functions (parabola, arctan, and so on).

When we come to column F, we've entered a different region. Just as the sine wave is 'familiar' yet often (usually?) misrepresented in casual representations (Figure **9**, page **19**), so the wheel seems familiar because it is 'a circle', but is generally misapprehended as to 'how it gets around'. At least in childhood, our naive concept of what a wheel does is this: 'It goes in a circle'. But really the wheel traces out a kind of 'jumping-kangaroo path' (Salas & Hille, p. 562) known as the *cycloid*. (Stick a blob of chewing gum near the edge of a dolphin-free tuna tin and roll the tin along a counter top. A cycloid will be traced in the air by the affixed blob.) That's the nonintuitive curve shown at the top of column F.[18] One of its distinctive features that is immediately noticeable is the presence of cusps. A cusp is one of the criteria cited in Figure **19** (**Chapter II: Limits**): okay for continuity but not for differentiability.

The cycloid is a close relative of the *brachistochrone*, *tautochrone* and *trochoid* (whose names mean least-time, same-time, and wheel-ish curve, respectively), all of them rather famous.

A Flatland Analogy? Not Quite

In our calculus 'trios', one may feel that there is something vaguely reminiscent of the relation between 3D/2D/1D shadow-projections, as sampled in Figure **33**.

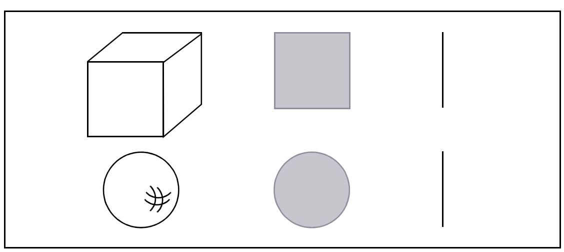


FIGURE 33: The Geometric Hierarchy as Notional Shadow Relations

In Figure **33**, the square and circle are intended as the 2D shadows of a cube and spherical ball. The vertical lines, in turn, are intended as the 1D 'shadows' or 'profiles' of the square and circle, all in oblique allusion to E.A. Abbott's *Flatland*. (These relations may in turn recall the shadow axis in Figure **24** and certain aspects of Figure **63**. Also, Priestley, pp. 38-40 may come to mind, although the relation there would be superficial since he is presenting 'Six Famous Ratios', fairly serious stuff, while I am only playing with some proposed metaphors.)

Nothing like the clear simple logic of these shadow relations will be found for the calculus trios in Figure **31** (except perhaps in columns A and B). However, if one invests, say, an hour learning my Curvature Kartouche (or its rather tedious and fragmentary academic equivalent), one will possess a Rosetta Stone that magically relates any two calculus curves that are adjacent in a given trio, one segment at a time.

Also, via that simple bit of arithmetic known as the Power Rule (page **75**), we will see that the first derivative of $V_{\text{ball}} = 4/3\pi r^3$ is identical to the equation for $A_{\text{sphere}} = 4\pi r^2$ (from middle school geometry), which is quite astonishing and certainly as 'clean' as the ball/circle/line relation shown in the lower half of Figure **33**.

Examples of how to use the Curvature Kartouche

The caption for each figure is where a pretend 'exam question' is posed. The answer is provided by call-outs in the figure itself and/or the ensuing text.

In Figures **34** through **37**, we keep using the same 'trio' that was introduced in Figure **25**. It consists of the function $y = 1/(1+x^2)$ (the arctan function from Table 5 on page 82), along with its first and second derivatives. All beautiful curves, I think, but that was not my only reason for choosing this particular function. It happens to be a function where the Inflection Points (IP) are nearly imperceptible to the eye. To find an IP, often you must look at a curve ***and*** apply logic — the logic of the kartouche, for example, which says you will find an IP in f wherever zero is found in f''. (The zeroes are easy to find visually, and they in turn lead you to the IP's.)

Since the same function is recycled several times, I do not label every feature of every curve, the way I did in Figure **25**. For example, in Figure **34**, I show the IP, the zero point and the 'flat' point on the right side of the graphic, simply for context, but on the left side I omit the corresponding labels as they would clutter the area where the 'action' is in this example. And so on.

It is important to realize at some point that the kartouche works 'all by itself'. Its logic is a world unto itself that doesn't need any of the curves that I've chosen arbitrarily to illustrate it. The reason I keep presenting the kartouche ***with*** curves is simply to build the reader's confidence that it works as advertised. At first, your attention will be largely on the illustrative curves, but with time you will 'throw them away' and look only at the kartouche, once you trust it. (I realize that this paragraph reads a little strangely, but once you've worked through one or two examples, you will see the point I'm trying to make here.)

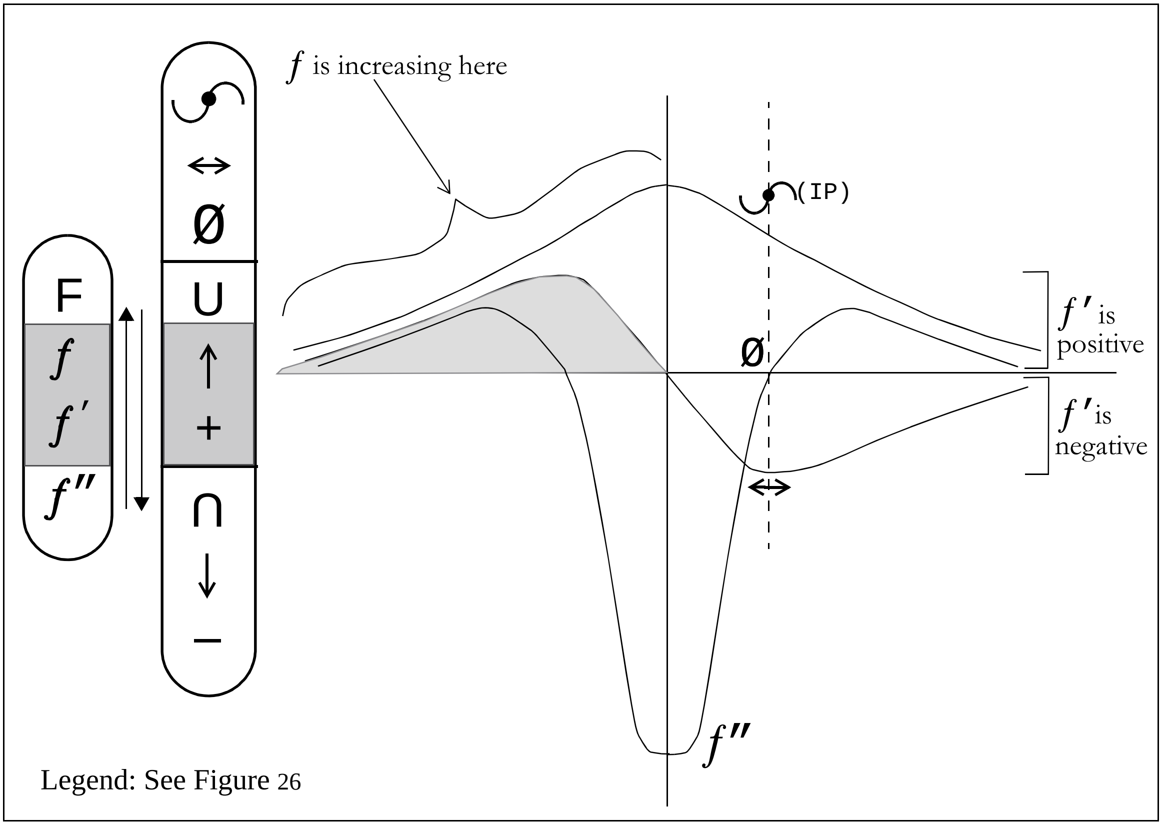


FIGURE 34: Where f is Increasing, is f' Positive or Negative?

Answer: f' is positive, as indicated by the shaded area above the x-axis.

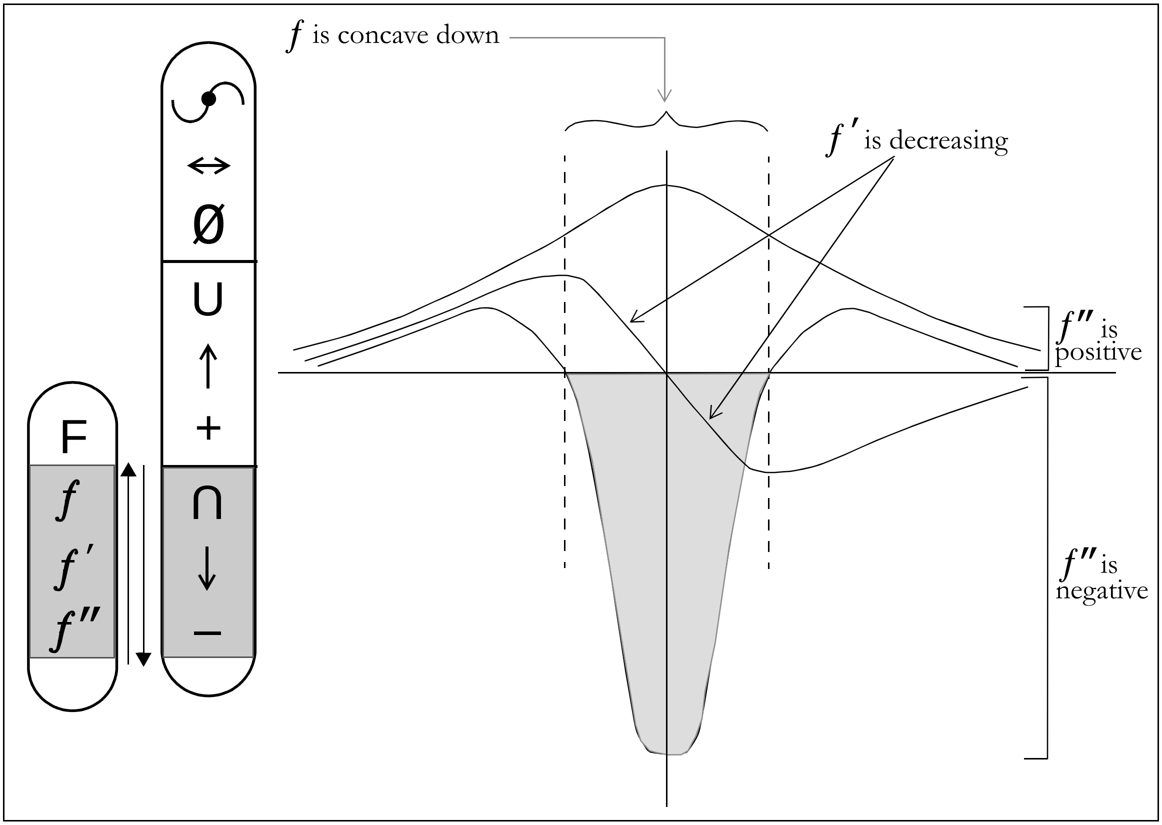


FIGURE 35: When f'' is Negative, What Are f and f' Doing?

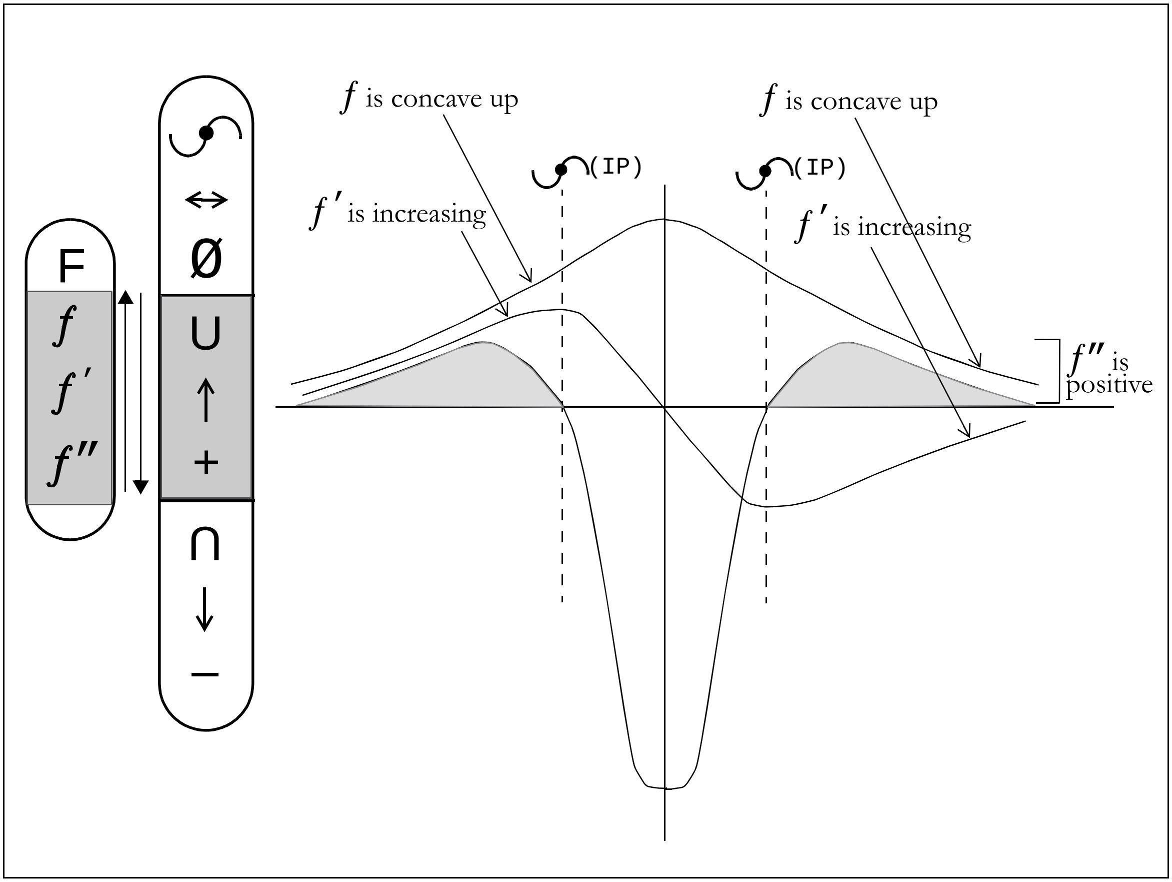


FIGURE 36: When f is Concave Up, What Are f' and f'' Doing?

This one is difficult to see. Its answer must rely partly on the graphic, partly on deductive reasoning: The part of the f curve that lies between the two IP points is clearly 'concave down'. Therefore, the segment to the left of the first IP (and the segment to the right of the second IP) *must* be 'concave up', even though the crudeness or subtlety of a given rendition may obscure this attribute at first.

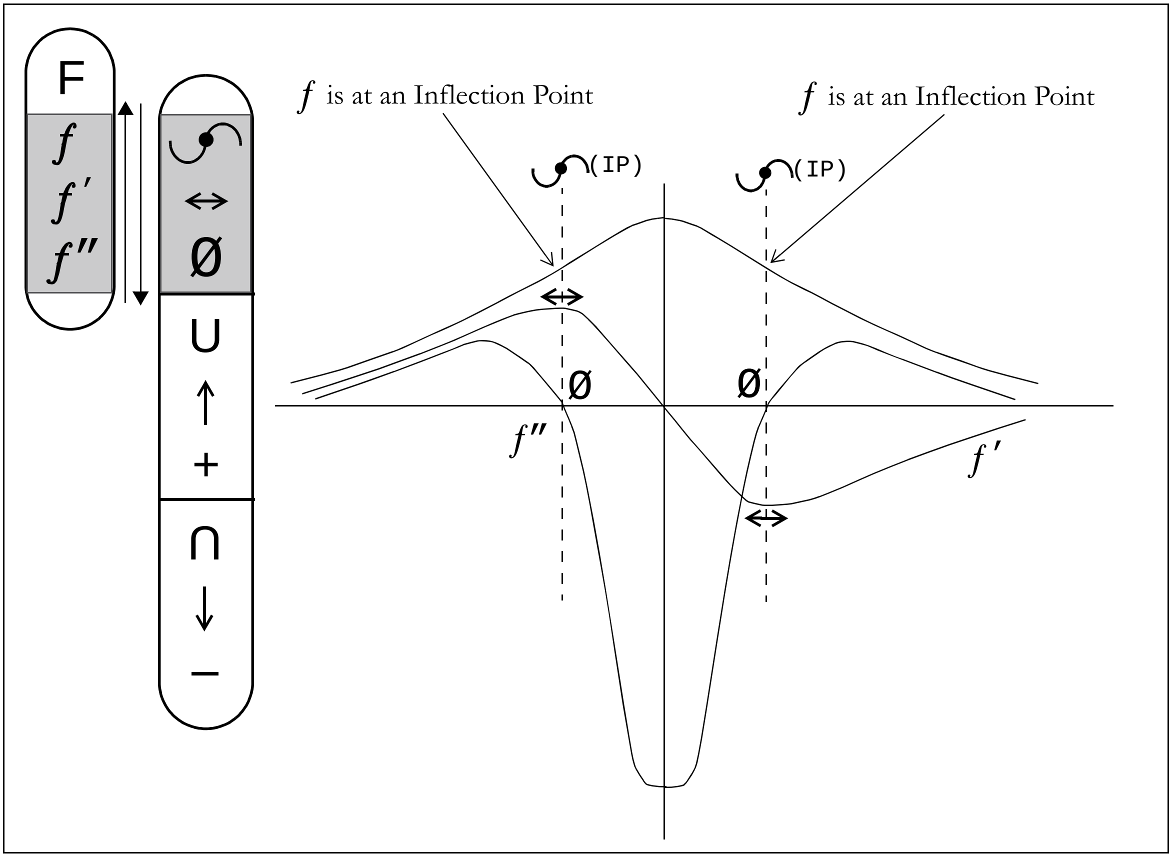


FIGURE 37: Where f'' is Zero, is f Flat or at an Inflection Point?

For Figure **37**, try 'reading your answer' to the question first from the graphic, then directly from the kartouche. Recall that the curves at the right are just 'training wheels' to be discarded at some point. The goal is to start reading directly from the kartouche as soon as you've seen it work in enough diverse cases that you begin to trust it.

V Integral Calculus

Introducing Integral Calculus: $\int$

In the heading above I say 'introducing' but really we have seen some examples of integration already: The shaded areas in Figures **20** and **24** are examples of integration. It's just that I have been avoiding the label 'integral calculus' as it sounds rather foreboding, freighted with history and connotations.

How to begin the formal introduction? Oddly enough, a little story about a broken speedometer and broken odometer might be just the ticket. In the second month of Calculus I, our instructor wrote something like this on the white board...

$R = \frac{D}{T}$ Rate as Distance over Time Differentiation: $\left(\frac{\Delta y}{\Delta x}\right)$	$D = RT$ Distance as Rate times Time Integration: $(\Delta y \bullet \Delta x)$

...accompanied by words to this effect: In the first case, imagine that you are driving somewhere, recording certain odometer readings and clock times in a journal, but your speedometer is broken. In the second case, imagine that the odometer is broken, so you take readings from the speedometer and the clock to 'get distance back'.

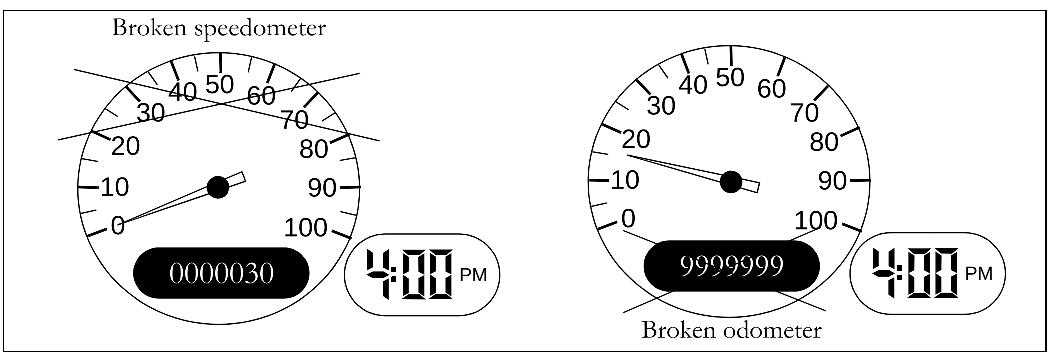


FIGURE 38: Playing with Rate, Distance, and Time

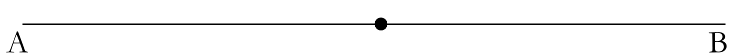
In Figure **38** I've sketched my impression of the two images that Dr. Naughton's words evoked.[19] In each image, one of the three instruments is broken, leaving only two to read. In the following pages, we will develop this automobile trip scenario using numbers I've selected so that our own speedometer/odometer/clock calculations can double as an exploration of the function $f = x^2$ and its antiderivative $F = x^3/3$, and ultimately as a reprise of the Fundamental Theorem of Calculus, in Figure **42**.

Everything that follows in this chapter is premised on an arbitrary data set consisting of paired times and distances, pertaining to an imagined automobile trip (admittedly a rather strange one, commencing evidently in the midst of a horrific traffic jam). See Table **1**.

TABLE 1: Clock and Odometer Readings

x	y
Time on Clock	Miles on Odometer
Noon	0
1:00 PM	1
2:00 PM	5
3:00 PM	14
4:00 PM	30
5:00 PM	55
6:00 PM	91
7:00 PM	140
8:00 PM	204

To begin, let's draw a picture of the driver in her car, say at 6:00 PM, roughly at the halfway point assuming A is zero and B is 204 miles (from Table **1** above):



The line is the road, the dot is her car, shown roughly at the 91-mile mark. The intent of this picture is to establish that the problem situation itself involves nothing 2-dimensional, nothing remotely concerned with *area*, only the movement of a car along a notional line (a straight road out of Beijing, if you like) at varying rates between points A and B. The problem itself can be modeled in one dimension. *Some* of our talk later of 'area under a curve' must therefore be an artifact of the calculus. (By contrast, in geometry, something like 'the area of triangle QRS' is not an artifact but the very *focus* of the discussion.) This business about n-dimensional modeling versus the n+1 dimension comes up repeatedly in calculus. See especially **Chapter VII**, where I try to pull it all together into a single bird's-eye view of the

Calculus III terrain.

The values from Table **1** are turned into a graph in Figure **39**. Suspended from each of the plotted points I've drawn a triangle. Each triangle represents R = D/T (using the triangle as a 'picture of division' as explained in **Chapter I: Slopes and Functions**). Each value for D is obtained by subtracting two consecutive odometer readings, e.g., 55 – 30 = 25 miles, and so on. Meanwhile, the value for T is always 1 hour. (This harks back to the triangle with base 1, likewise introduced in **Chapter I**.)

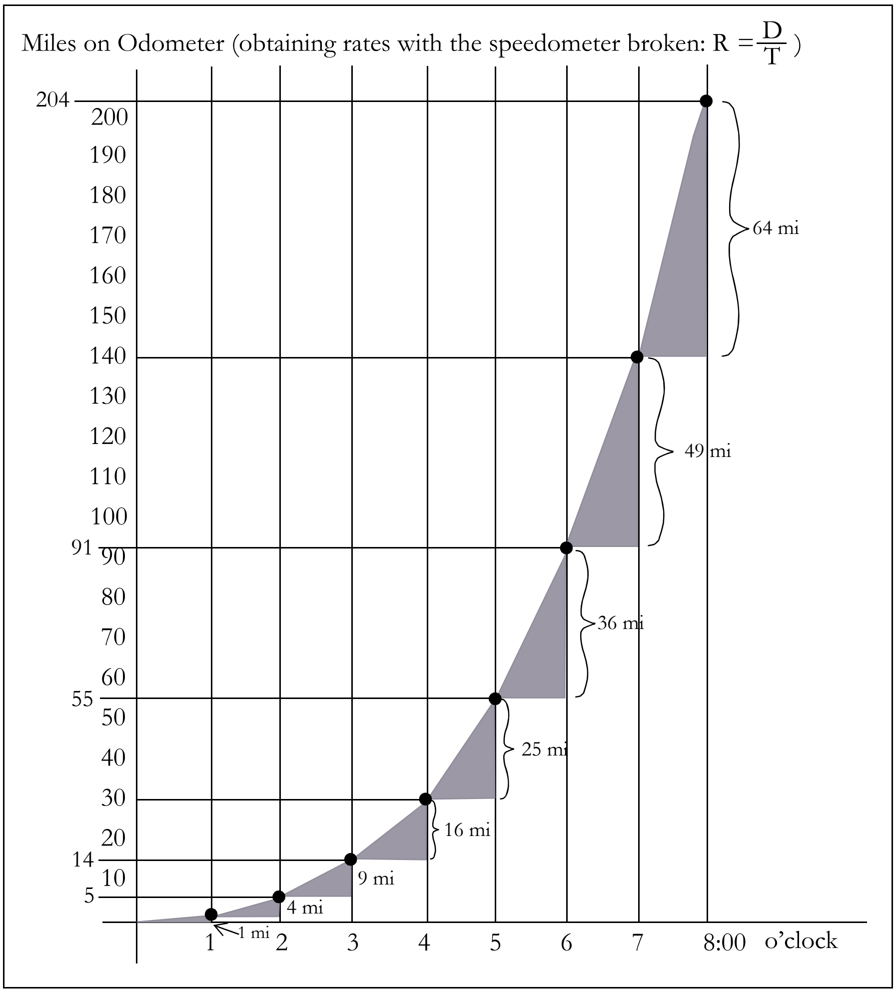


FIGURE 39: Odometer and Clock

Before leaving Figure **39**, note in passing the illusion of a *curve* where all I've done is string together a series of five graceless *hypotenuses.* (That wasn't my intention, but it is fun to contemplate, and yes it is pertinent to calculus, which in one sense is the language of curves, as celebrated in **Chapter IV**, but is also the field that paints all curves as possibly illusory, just a succession of stubby tangents. Your choice.)

In Figure **40**, we imagine a second person driving immediately behind the first one. Because of traffic conditions, the second car mimics exactly the progress of the first car (i.e., its different speeds at different times of the journey). But on this dashboard it happens that the *odometer* is broken, so the handwritten journal for this car's trip is kept in terms of speedometer readings and clock readings.

This time each point plotted is accompanied by a rectangle to its left. Each rectangle represents a calculation of D = R • T. (Here we have the working out of $\Delta y \bullet \Delta x$ from page **63**. For more perspective on that expression, see the entry for $\int$ on page **196** and note **19** on page **236**.) Since the time interval for a given rate is always one hour, the ' • T ' part of the formula would seem to be a nonevent, except that it carries the unit 'hr' to be cancelled out each time:

D = 1 mi/hr * 1 hr = 1 mi
D = 4 mi/hr * 1 hr = 4 mi
D = 9 mi/hr * 1 hr = 9 mi
D = 16 mi/hr * 1 hr = 16 mi
etc.

Each of these values matches the label on one of the triangles in Figure **39**, and they sum to 204 miles, in agreement with Table **1**. Thus, by integrating, one has been able to 'get distance back' (as our Calculus I teacher enjoyed saying, just to add a bit of metaphorical edge or poetry to the whole, we felt). Perspective: This has been an example of *direct* integration, nothing to do with the FTC. As represented in Figure **40**, the method looks rather crude. The polished version of this method is called *Riemann sums* (see page **218**, also the entry for 'integral sign' on page **196**).

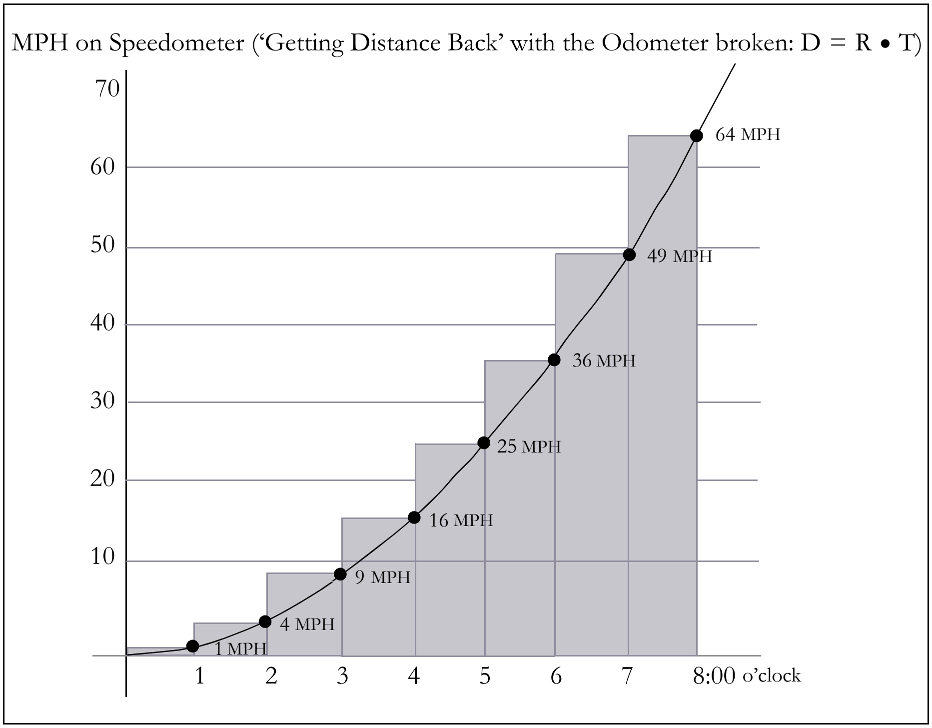


FIGURE 40: Speedometer and Clock

Now we look at Figure **40** from a different perspective. As mentioned earlier, I chose numbers for Table **1** that would lead us eventually to the graph of $y = x^2$. In Figure **41**, I've replicated the curve from Figure **40**, extending it slightly to make it look more parabolic (since that is the shape traced out by $y = x^2$, a parabola).

Let's try treating this parabola as a derivative function, and see if it possesses a useful antiderivative. In other words, let's integrate. This is where the notation convention is tricky. We write f under the integral sign: $\int f$ (abbreviated from Figure **42**). But *this* f is not *that* f, the one you may be thinking about. Under the integral sign, it no longer means the *function* f. Rather, it means the *derivative* function f (corrresponding to some antiderivative F, which in this instance is yet to be discovered).

To discover the antiderivative, we apply the Power Rule to x^2. That means incrementing the exponent, then dividing the result by the new exponent, which is to say $F = x^3/3$. (Here I've jumped slightly ahead of myself, since the formal presentation of the Power Rule doesn't come until page **75**. My apologies for the zigzag.) Why do we care about this antiderivative, $F = x^3/3$? Because it will allow us to obtain a much better estimate of area under the curve than by summing the rude rectangles of Figure **40**. An example: Suppose we want to know the area under the curve on the interval from 4 o'clock to 6 o'clock. Using the rectangles as drawn in Figure **40**, we would estimate the area as 25 + 36 = 61, i.e., those two rectangles represent a distance of 61 miles traveled during that time period. (Recall that I said I would draw rectangles to the *left* of each plotted point. That's why I select the values 25 and 36 for this exercise, not 16 and 25. Alternatively, I could have drawn the rectangles to the right of each plotted point, in which case everything would be understated instead of being overstated. Not better, just wrong a different way.) Using the antiderivative instead, and applying the FTC, we would calculate the area as 51, as shown in Figure **42**. Where is the antiderivative in Figure **41**? I show only part of its curve, as a dotted line, since visually it does not play especially well with the parabola. For the most part, the antiderivative is represented by the dark vertical bar running from 21 to 72 along the y-axis.[20]

By subtracting one of those values from the other we obtain 51, a much better representation of the area than 61 earlier, which substantially overstated the true value, as one can see simply by glancing at Figure **40**. (Quick check on the calculus result: Eyeballing the area in question, we can see that it is comprised of a 16 by 2 rectangle, topped by something that looks very close to a triangle with height 20 and base 2. Adding together the areas of those two shapes, we have $32 + 20 = 52 \approx 51$. Apparently, our strange looking antiderivative, $x^3/3$, is working as advertised! Little thrill?)

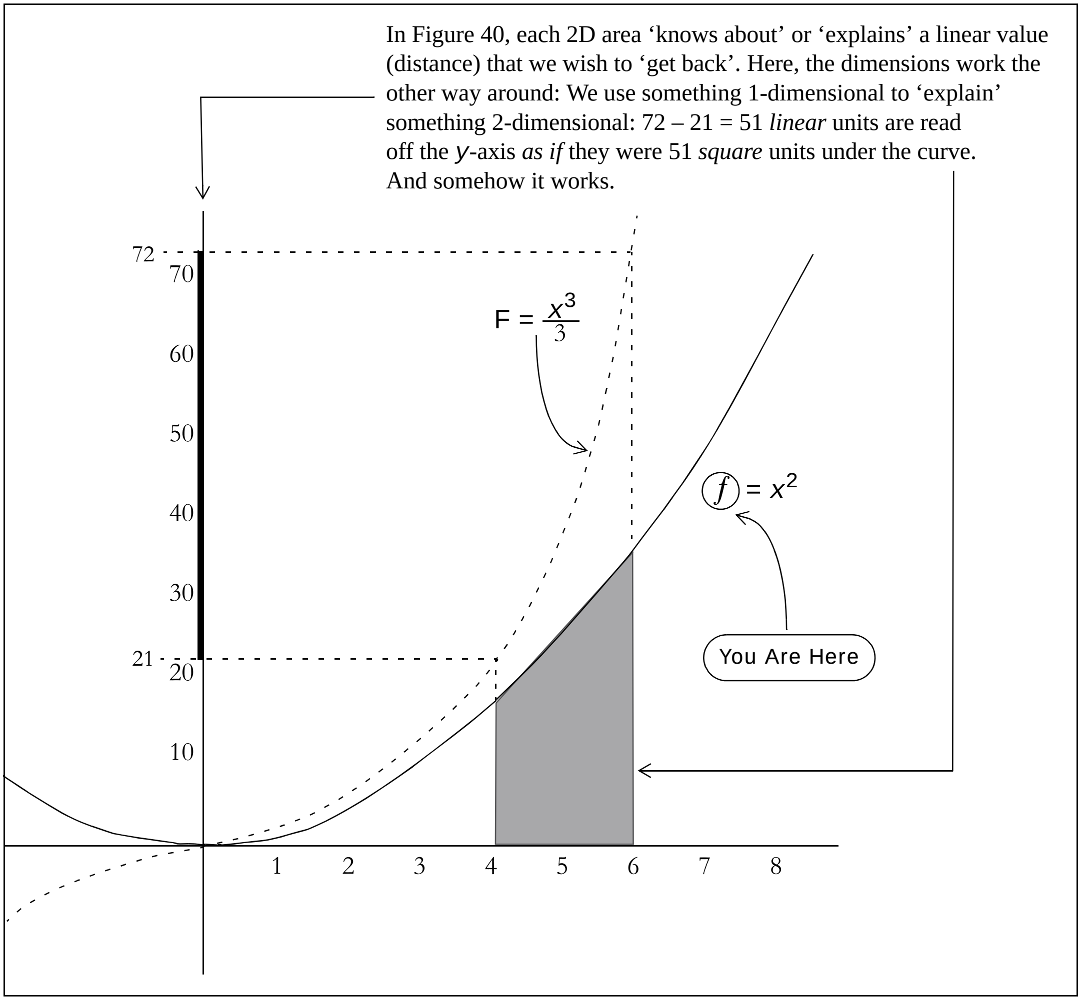


FIGURE 41: A Second Look at 'Area Under a Curve'

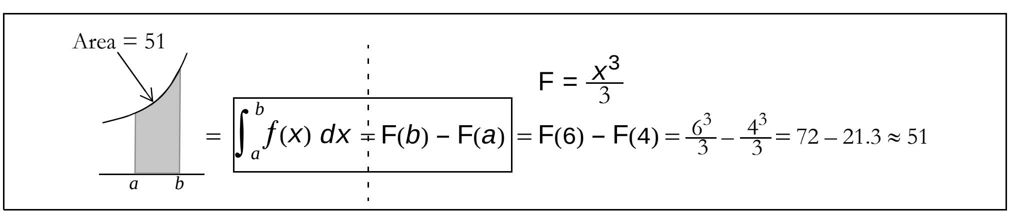


FIGURE 42: Reprise of the FTC (from **Chapter III**)

The other point I wish to make may strike you as somewhat ephemeral, but it is well worth attempting it I think. On page **64** I remarked that some of our talk later of 'area under a curve' must be an artifact of the calculus. In this narrative about the automobile trip, when we first encounter Figure **40**, the second dimension has been conjured seemingly out of nothing. After all, the problem was modeled as a *dot* on a *line*, yet here we are adding up *rectangles* under a *curve*. I think it is fair to characterize this activity in the $n + 1$ dimension as an artifact of calculus.

But there is no law that says we cannot initiate a closely related problem in a planar region that is real. In fact, that's what we did next. We said, in effect, "Here is a region that I have shaded with a pencil on a physical sheet of paper. Now tell me what its area is." No longer was the planar region a phantom. And from there we invoked the Power Rule to discover the function's antiderivative, which allowed us to calculate the shaded region's area (and to do it better this time than with our clumsy set of eight rectangles). Thus, calculus may be said to live in a rather mysterious realm that straddles the real and the unreal. The second dimension as it appeared in Figure **40** was not quite real, even though it helped us with something real, a first approximation of the missing odometer readings. And in that sense integration as a tool might remind one of the 'imaginary number' which has surprisingly down-and-dirty practical applications, as documented in the book by Nahin that is devoted to i.

Multiple Integrals

The generic term for an integral that has this general appearance...

$$\overbrace{\int_c^d \underbrace{\int_a^b f(x,y)\, dx}_{\text{inner integral}}\, dy}^{\text{outer integral}}$$

...is *multiple integral*.

Since one integral is nested within the other, *I* would favor calling this a *nested integral* (borrowing from the vocabulary of the computer programming world). But the point is moot since we are only discussing generic name(s) so far. Moving on to the *non*generic terms, we have the following specific ones to consider, which are more of a technical nature:

- *double integral*: an integral involving both dx and dy. (This kind typically has two integral signs as shown above; however, it may also be written with a *single* integral sign, as illustrated later in this section.)
- *iterated integral*: an integral involving both dx and dy (canonical order) or dy and dx. (This kind always has two integral signs. Variation on the theme: $d\theta$ and $d\phi$. For an example see page **146**.)

At this point, an excerpt from Stewart will help clear the air and set the stage:

> The evaluation of double integrals from first principles is even more difficult [than for a single integral], but in this section [16.2] we see how to express a double integral as an iterated integral, which can then be evaluated by calculating two single integrals. — Stewart, p. 1010

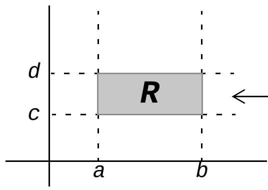
Saying it another way, the double integral is a real (and difficult) *type* while the iterated integral is more of a *technique* (rather fun and easy) than a type. In doing an iterated integral, while evaluating the inner integral with respect to x, one holds y fixed. It is natural to call this *partial integration*, by analogy with partial differentiation (page **90**). As is often the case, these concepts are easier to learn with an example than from a verbal description. So don't worry if this whole page looks like gibberish. Rather, forge ahead. Work all the way through the ensuing example, and the bullets above should start to make sense.

To keep things manageable, we will look at the simplest imaginable multivariable function, $f(x,y) = x + y$, and see what it does within the confines of the simplest bounds of integration, 0 to 1 along each axis.[21] Even when following this minimalist ethos, a rather wonderful little picture pops up on the grid. (Or, if one is determined to be churlish, one could say instead that "In calculus, 'nothing simple is simple'.")

Expressed in generic terms, the iterated integral notation looks like this...

$$\int_R f(x,y)\,dx\,dy \quad \text{or} \quad \int_c^d\int_a^b f(x,y)\,dx\,dy$$

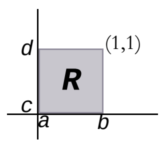
...where R or a,b,c,d refers to a region such as the following shaded rectangle:



$$\longleftarrow \; R = \{\,(x, y) \mid a \le x \le b,\ c \le y \le d\,\}$$

Now the specific problem.
The function we've chosen to evaluate over the interval [0, 1] is $f(x,y) = x + y$.
Using the option for a single integral sign, our double integral is

$$\int_R (x+y)\,dx\,dy$$



$$\longleftarrow \; R = \{\,(x, y) \mid 0 \le x \le 1,\ 0 \le y \le 1\,\}$$

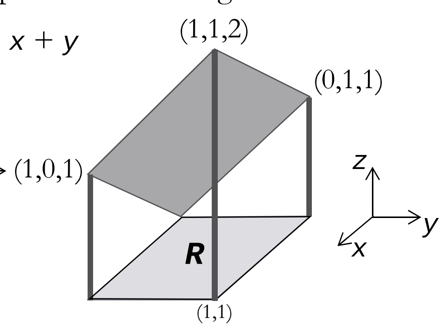
When it is rewritten as an iterated integral, we have this instead to evaluate:

$$\int_0^1\int_0^1 (x+y)\,dx\,dy$$

But first, just for fun, let's use the function to plot some values of x and y, and thus find the coordinates of the implied solid *over* region R.

$$z = f(x,y) = x + y$$

x	y	f	As xyz coordinates
0	0	0	(0,0,0)
1	0	1	(1,0,1)
0	1	1	(0,1,1)
1	1	2	(1,1,2)



(In connection with the notion of an 'implied solid', see **Implicit y, Implicit z, Implicit w** on page **134**.)

Now we know what it looks like: An exotic dwelling where three tall stakes support an awning that has one corner anchored to the ground. (That is the (0,0,0) point, which I leave unlabeled above to reduce clutter.) The question is, how do we calculate the volume of such an irregular space?

Here is the integral to be evaluated, repeated from the previous page, now stretched slightly in the horizontal direction, to accommodate the annotations:

DO THE INNER INTEGRAL FIRST
DO THE OUTER INTEGRAL SECOND

$$\int_0^1 \int_0^1 (x + y)\, dx\, dy$$

By the Power Rule (p. 75), an implicit x^0 has been integrated to explicit x^1.

Here, y has been held constant (not yet 'processed') since this is *partial integration*.

$$\int_0^1 (x + y)\, dx = \frac{x^2}{2} + yx \Big|_0^1 = \left[\frac{1}{2} + y\right] - \left[0\right] = \frac{1}{2} + y$$

$$\int_0^1 \left(\frac{1}{2} + y\right) dy = \frac{1}{2}y + \frac{y^2}{2} \Big|_0^1 = \left[\frac{1}{2} + \frac{1}{2}\right] - \left[0\right] = \boxed{1}$$

By the Power Rule, an implicit y^0 has been integrated to explicit y^1.

Conclusion: Severely askew though the 'awning' may be, the volume beneath it turns out to be an even 1 cubic unit. Visual check: The room's dimensions appear to be 1 by 1 by something greater-than-0-but-less-than-2. This is close enough to 1 x 1 x 1 that the calculated answer seems plausible.

Comment: Once we've machet-éd our way past the exotic notation and nomenclature, down to the *thing itself*, partial integration (via iterated integrals as demonstrated above) turns out to be one of the most fun and easy topics in the whole calculus curriculum. There are plenty of things that the student should worry about in calculus, but *this* is not one of them! (Likewise partial differentiation with ∂. Same misapprehension.)

VI Rules

Part I: Differentiation/Integration Rules as Matched Pairs

Well, they're called 'rules' but each of these is actually a 'tool'. Seeing how long this chapter is, you might feel better about it if you think of them that way: a big collection of tools (or rather, a modest *sampling* of the many tools available for doing calculus!)

The Power Rule

Typically one is presented with a set of half a dozen rules in Calculus I, all pertaining to differential calculus, then in Calculus II one is presented with another set of rules, all pertaining to integral calculus. Strictly speaking, the integration rules might be sampled or previewed toward the end of Calculus I (the example on page **63** is based on such a preview as provided by my own Calculus I instructor), but still it feels as though they are essentially a Calculus II event since the Integration Gateway Test occurs in Calculus II. Be that as it may, the two groups of rules are *segregated*, that's the salient point.

Now the second set of rules turns out to be the first set turned backward. And I think it does no harm to have this 'big picture' in mind from the outset — the idea of a parallel set of rules for running forward (from function to derivative) or backward (from derivative to antiderivative, alias the original function). Accordingly, for the first half-dozen cases, starting with the Power Rule in Figure **43**, I will be introducing both aspects together — both 'forward' *and* 'backward', so to speak.

True, the traditional approach in which the forward rules and retrograde rules are segregated provides modularity, which is convenient for both students and teachers in planning out their academic years. But from our viewpoint here in the 'oasis', it looks a bit strange. A rough analogy: Suppose fire truck extension ladder training were structured as follows: On Monday, one learns how to climb with heavy equipment up to the top of the ladder, then a helicopter whisks you away at the end of the session. On Tuesday, the helicopter deposits you back at the top of the ladder, and now you learn how to climb back down the extension ladder, carrying heavy equipment on your back. Is it reasonable? Why not just learn how to climb and descend the ladder all in one training session?

(In Part II and Part III of this chapter, when we come to the Chain Rule and its counterpart Integration By Substitution and the Product Rule and its counterpart Integration By Parts, I relent and present these four independently, not by pairs, since each of them seems to possess a critical mass of its own.)

The power rule comes first because it is arguably the most important of all the rules. The rule itself is short, but it provides an opportunity to understand some of the notational problems mentioned in the **Prologue**, and for that reason I will give a rather extended exposition of the rule (or rather, pair of rules, forward and backward). For context, it would be good to revisit Figure **6** on page **16** at this point.

In Figure **43**, I show my own proposed presentation scheme first, followed by the conventional one (the way your instructor would probably introduce these rules on the white board). Seen in isolation, the conventional presentation looks reasonable. Seen in the context of my proposed presentation, it begins to exhibit some peculiar shortcomings, I think.

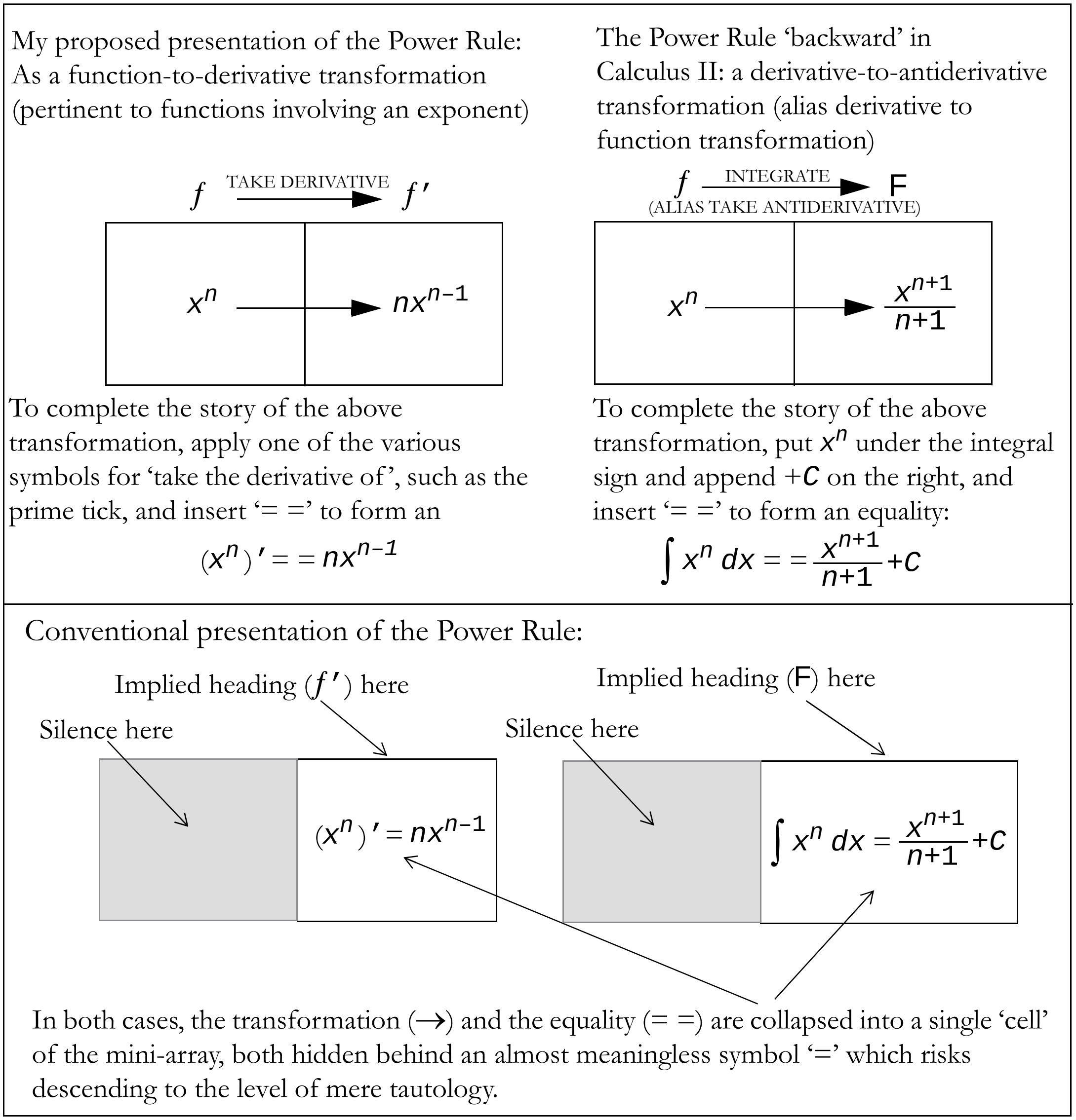


FIGURE 43: The Power Rule, My Way and the Conventional Way

Comments on Figure **43**: What the student is trying to do is acquire the rules for two kinds of *transformation*. My presentation method shows exactly what the two transformations are. (For a discussion of the symbols '→' and '= =', please refer to page **200**f.) Sphynx-like, the math establishment offers, instead, two *static* pictures of the situation after the fact: A pair of cheat-sheet mnemonics that would be useful, yes, to someone who is already fluent in the use of the rules, but to a student? Once acquired, the Power Rule is one of the more fun and easy parts of calculus, but thanks the conventional presentation, it has the look-and-feel of something recondite at first, not the simple arithmetic operation that it is. (For a slightly different angle on the Power Rule, see Table 10 on page 226.)

Example of the Power Rule used in the forward direction, for differentiation (i.e., to discover the derivative of a function):

$$f = x^2$$
$$f' = (x^2)' = 2x^{(2-1)} = 2x^1 = 2x$$

Example of the Power Rule used in the retrograde direction, for integration (i.e., to discover the antiderivative of a derivative function f):

$$f = 2x$$
$$F = \int 2x\,dx = (2x^{1+1})/2 = (2/2)x^2 = x^2$$

Thus, we come full circle to x^2, as advertised.

(If you are comfortable with software engineering lingo, you might prefer to express the forward rule this way: 'prepend exponent, decrement exponent'. That's my own preferred wording. Granted, 'prepend' is not exactly a word, but it should be, as any computer geek will tell you.)

The '+ C' in Figure 43 stands for 'plus any constant'. I.e., you may append +3 or +777 etc. and still have a 'correct answer'. This peculiar circumstance may be described as a side-effect of the Constant Rule, next. (For an example of the Power Rule in context, revisit Figure 40 on page 67 and the discussion of $F = x^3/3$.)

Constant Rule and '+C'

Wherever a function contains a naked number, that's a constant (opposite of a variable), and the derivative of any constant is 0. That's the rule. Stated in its most abstract form, we have $C' = 0$ (using prime notation to mean 'take the derivative of...'). Here are two concrete examples: $(6)' = 0$, $(21)' = 0$. In pictures, the slope of function $y = 6$ is flat (zero), likewise the slope of $y = 21$ is zero.

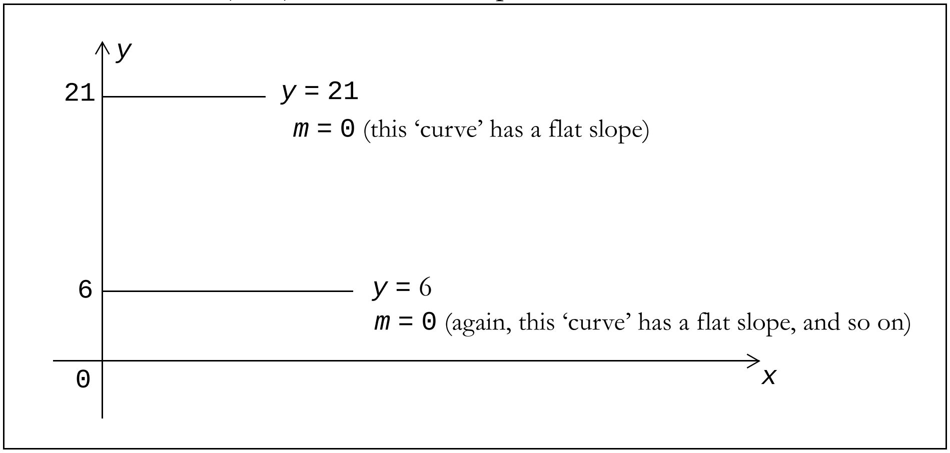


FIGURE 44: Two Functions With Flat Slope (Zero Slope)

Or, placing '6' in a more realistic context, if $y = x^2 + 6$, then the derivative of that function is calculated as follows: $y' = [x^2 + 6]' = 2x + 0 = 2x$. The Power Rule changed x^2 to $2x$; the Constant Rule changed 6 to 0, which was then discarded as useless.

Now, knowing that 6 has vanished into the beyond, how can we hope to 'get the constant back' and make 6 reappear when we integrate zero? Of course $\int 0 = ?$ is a hopeless proposition (if it is three months later, for instance, and you have no recollection that zero came from 6 and not from 21 or a million). What to do? The wily mathematician is not so easily routed. Brazenly, she writes $\int 0 = C$, meaning "the antiderivative of 0 is some constant, but I may never know its value." Crisis averted! Reversing the example above, if we wish to *start* with $2x$ and integrate it, we write the following:

$$\int 2x = x^2 + C$$

The Power Rule in reverse turns $2x$ into x^2. But where does '+ C' come from? This is the flip side of the Constant Rule: No matter what, every single integration concludes with '+ C', *just in case* some constant went into the integrand's making. That way, we look smart, cognizant of all those vanished Cheshire Cats haunting the void. Better yet, it permits us to understand what the teacher means when she says, "It's okay to be off by a constant." Translation: "Everything before '+ C' is what counts. The '+ C' itself is a throwaway. Deep down we know we'll probably never be able to assign it a value. Moreover, there may very well have been *no* constant at all in the original function to 'get back'. So let's not worry about it." (Note the close resemblance to the technique of conjuring a hidden '1' and integrating it to 'x', as described in connection with Integration By Parts on page **99**. Here, we conjure a *maybe*-hidden '0' in the integrand and integrate it to C-*as-insurance*. Same technique, with a slightly more devious, legal-flavored intent.)

Also, knowing about '+ C' helps you understand why we see mainly references to '*an* antiderivative of the function f' and only rarely a ('wrong') reference to '*the* antiderivative of the function f'. Outside the walls of academia, clearly there *does* exist an entity that is *the* antiderivative of the function f, but the convention is to honor the (possible) presence of an important constant hiding behind the mask of '+ C' and therefore say 'an' instead of 'the'. For more about this, see my semi-facetious definition of ∞ on page **216**.

Log Rule (alias Ln Rule)

TABLE 2: Log Rule, Forward and Backward

	LOG RULE
Forward (for differentiation in Calculus I)	$(\ln x)' = \frac{1}{x}$
Backward (for integration in Calculus II)	$\int \frac{1}{x} dx = \ln x + C$

This is a tricky one. Sitting off by itself, the expression $1/x$ looks like a natural for application of the Power Rule, but that way lies disaster. To integrate $1/x$, write $\ln x\ (+C)$. In other words, if you ever find yourself incrementing x^{-1} to x^0, in an attempt to apply the Power Rule, a little red flag should pop up. Other considerations: Only if the whole denominator = w, then $\int 1/w = \ln |w|$. (I've notated this example with the assumption that we're in the midst of some sort of w-substitution. But why the vertical bars? Strictly speaking, it is the *absolute value* of w that is the argument of the $\ln$ function. Neglect of this nuance is another frequent source of student error.) Conversely, if w has a radical or power, like $\int 1/\sqrt{w}$, then the Power Rule *does* pertain, not the Log Rule (alias Ln Rule).

Exponential Rules

TABLE 3: Exponential Rules

EXPONENTIAL RULES	Special	Generic
Forward (for differentiation in Calculus I)	$(e^x)' = e^x$	$(a^x)' = (\ln a)a^x$
Backward (for integration in Calculus II)	$\int e^x\,dx = e^x + C$	$\int a^x\,dx = \frac{a^x}{\ln a} + C$

As indicated in Table **3**, e^x 'is its own derivative'. For more about this, see **Appendix E**.

Trig Rules

TABLE 4: The Trig Rules, Forward and Backward

The Six Derivative Formulas ('forward')	
$(\sin x)' = \cos x$	$(\csc x)' = -\csc x \cot x$
$(\cos x)' = -\sin x$	$(\sec x)' = \sec x \tan x$
$(\tan x)' = \sec^2 x$	$(\cot x)' = -\csc^2 x$
The Six Integration Formulas (for reversal of the derivative formulas)	
$\int \cos x\,dx = \sin x + C$	$\int \csc x \cot x\,dx = -\csc x + C$
$\int \sin x\,dx = -\cos x + C$	$\int \sec x \tan x\,dx = \sec x + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \csc^2 x\,dx = -\cot x + C$

Some of the trig rule 'reversals' are nonintuitive. In particular, if one juxtaposes the derivative of $\sin\theta$ with the derivative of $\cos\theta$ the pattern seems asymmetrical. As a consequence of this apparent asymmetry, the student often stumbles when taking the derivative of $\cos\theta$ or taking the antiderivative of $\sin\theta$. (These two little items make a disproportionate contribution to students' overall errors in elementary calculus.) But seeing these formulas together, in the context of the full table helps tame them, I think. In Table **4** note the two triangles formed by the six formulas that involve minus signs. So there is a kind of symmetry after all, visible only from this higher vantage point.

Executive Decision on Notation: In Table **4**, why do I write '$\sin x$' not '$\sin\theta$'?

The point is recitation *and* 'scribbling' of the rules: The latter form with *theta* is too clumsy for recitation, also too awkward for when you wish to review the rules by writing them out, at a fast scribbled pace. When the time comes to play back one of these expressions in homework or on an exam, you will remember that really 'x' means 'θ'.

Regarding '+ C', please refer to the discussion of Figure **43** above.

TABLE 5: The Trig Inverse Rules

Calculus I: Simplified Version of the arcsin and arctan Rules	
arcsin $\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C$	arctan $\int \frac{1}{1+x^2}dx = \arctan x + C$
Calculus II: The 'Real' arcsin and arctan Rules	
arcsin $\int \frac{1}{\sqrt{a^2-w^2}}dw = \arcsin\left(\frac{w}{a}\right) + C$	arctan $\int \frac{1}{a^2+w^2}dw = \frac{1}{a}\arctan\left(\frac{w}{a}\right) + C$

Typically, in Calculus I one is introduced to two or more Trig Inverse Rules as represented in the upper half of Table **5**. In Calculus II one encounters them again, now in the more versatile form shown in the lower half of the table. For some representative examples of such a function, see Figures **34** through **37** in **Chapter IV**, where I repeatedly exploit the arctan function and its first two derivatives. See also the Trig Identities on page **165**.

Not covered in this book:

- Hyperbolic Trig Rules,
- Trig Substitution.

Part II: Differentiation Rules Continued (not as matched pairs)

Product Rule and Quotient Rule

There is really no way to make the Power Rule look pretty (in its formal statement, on page **75**), even though conceptually it is straightforward and easy to learn. By contrast, the Product Rule, *if* properly notated, is one of the most elegant rules in calculus, engaging even in its formal statement:

$$(fg)' = f'g + f\,g'$$

Unaccountably, in many books it is given using notation that turns it into a hideous and unwritable thing. Here is one of several variations on the theme:

$$\frac{d(uv)}{dx} = \frac{du}{dx}\bullet v + u\bullet\frac{dv}{dx}$$

That's the style I call 'dove notation', in honor of its 'duv' combinations, although 'ugly enough to scare little children' might be a more judicious name for it. It is interesting to note that the sane one and the hideous one are found together in Hughes-Hallett *et al.*, on page 122. (For once, their committee from hell did something right. Credit where credit is due!)

The Product Rule is reversed by Integration By Parts (page **95**).

As one would expect, the Quotient Rule bears a close resemblance to the Product Rule:

$$\left(\frac{f}{g}\right)' = \frac{f'g - f\,g'}{g^2}$$

Or if one has acquired a taste for 'dove' notation, then:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}\bullet v - u\bullet\frac{dv}{dx}}{v^2}$$

It is, again, Integration By Parts (page **95**) that reverses the Quotient Rule (by way of a slight variation on the method employed to reverse the Product Rule).

Chain Rule: Single Variable

There are several kinds of chain rule. The chain rule first makes its appearance in Calculus I, in connection with composite functions. Purpose: To differentiate a function that seems to mix two or more function types. (Compare **Product Rule**

and Quotient Rule above, where similar but distinct problems are addressed.) In prime notation, the chain rule looks like this:

$$h(x) = f(g(x)) \qquad h'(x) = f'(g(x)) \cdot g'(x)$$

Usage:

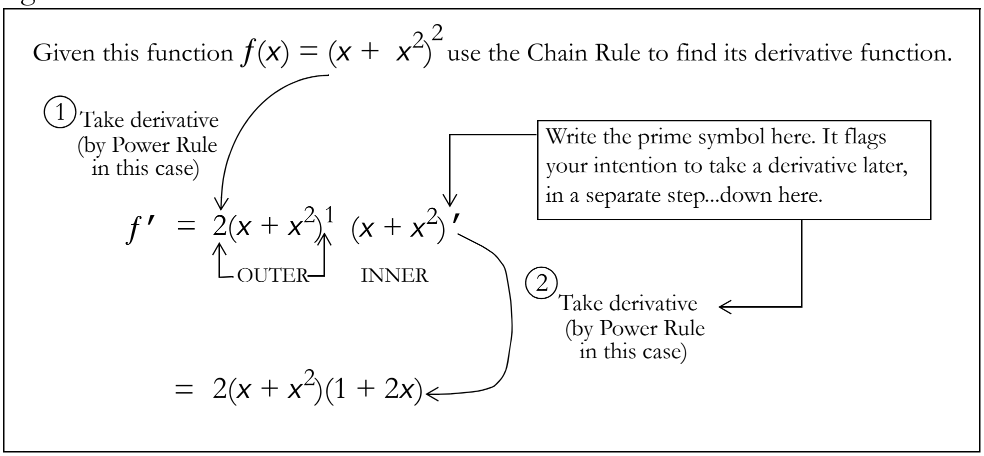


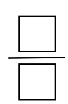
FIGURE 45: The Chain Rule with Single Variable

This is only the tip of the ice berg — but it is a carefully chosen 'tip' — just what you need to know about the chain rule with a single variable to survive Calculus I. For another important installment of the story, please refer to **Appendix B: The Chain Rule(s)**.

Chain Rules: More Than One Variable

In Calculus II, other kinds of chain rule make their appearance, notably the one I call 'Classic Chain' and its cousin, the 'Twisted Chain'.

The Classic Chain works like an exotic form of algebra, built upon an empty fraction or double-decker box...



...instead of the variable 'x'. It is best introduced by way of an example:

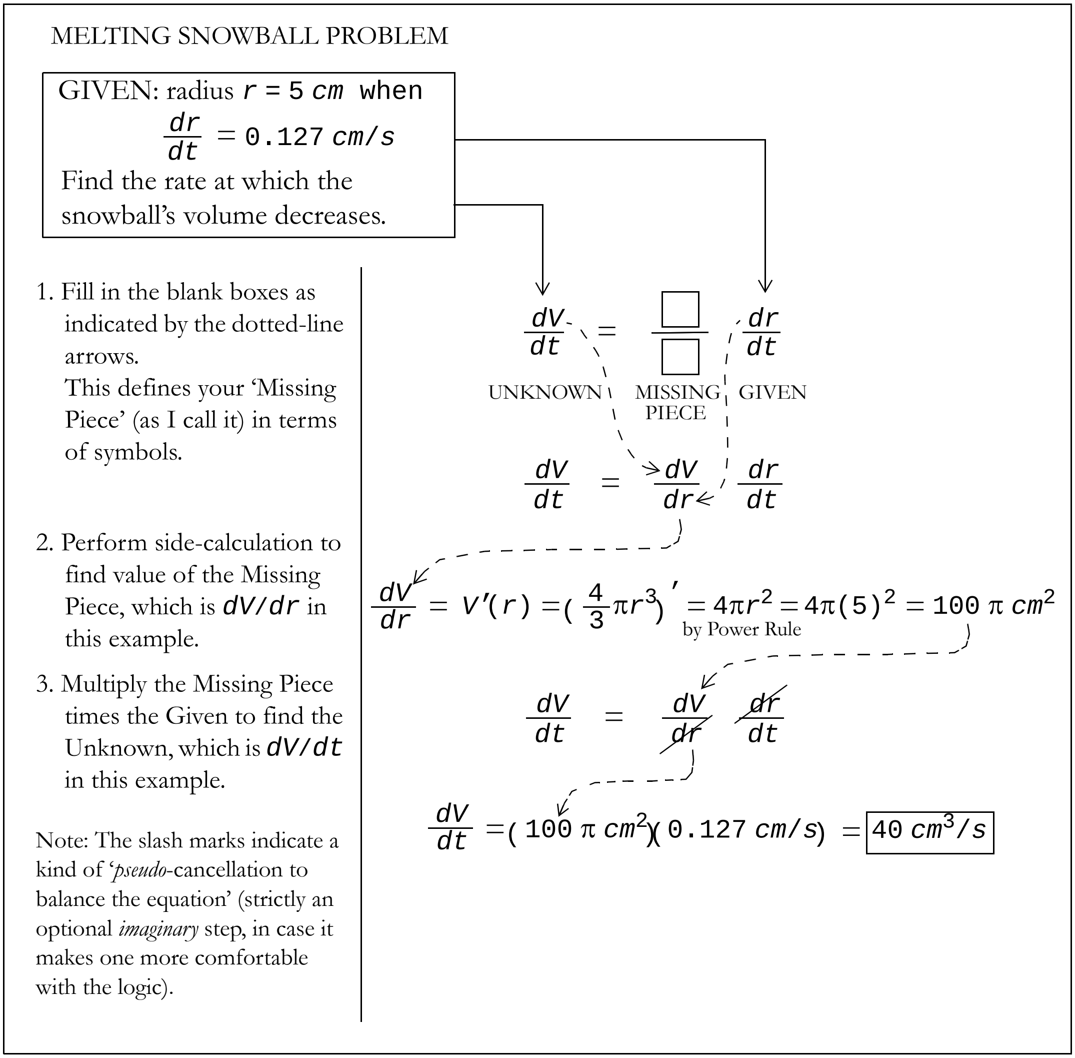


FIGURE 46: Classic Chain Rule: Melting Snowball Derivative

Comments: [1] The version of the chain rule illustrated in Figure **46** may be described as a 'passive bystander': 90% of your work involves dV/dr (dotted lines). [2] Note that the rate $40\ cm^3/s$ is *instantaneous*: valid only for the moment when $r = 5$ and $dr/dt = 0.127\ cm/s$, as given at the outset. Understandably, textbook authors often tire of spelling out the instantaneous conditions for this kind of problem, hoping that they will be 'understood' by the student, by context. For instance, the givens might have been stated as follows...

$$r = 5,\ dr/dt = 0.126\ cm/s$$

...without the pregnant word 'when', leaving the student to supply it. There is a good discussion of this subtlety (including the somewhat unrealistic case where the author might include a constant rate among the givens) in Hughes-Hallett *et al.*, p. 206-207.

Twisted Chain Rule (Figure **47**)

This next version is a takeoff on the Classic Chain Rule (as I call it) that involves some convoluted logic. Strange though it looks, the Twisted Chain Rule exists for a very good reason, and that's why I devote space to it here in the body of the text rather than in an appendix. (Granted, to a math major, it probably just registers as 'a trivial variation' on the one I call Classic Chain Rule, but to my way of thinking it demands a special explanation.)

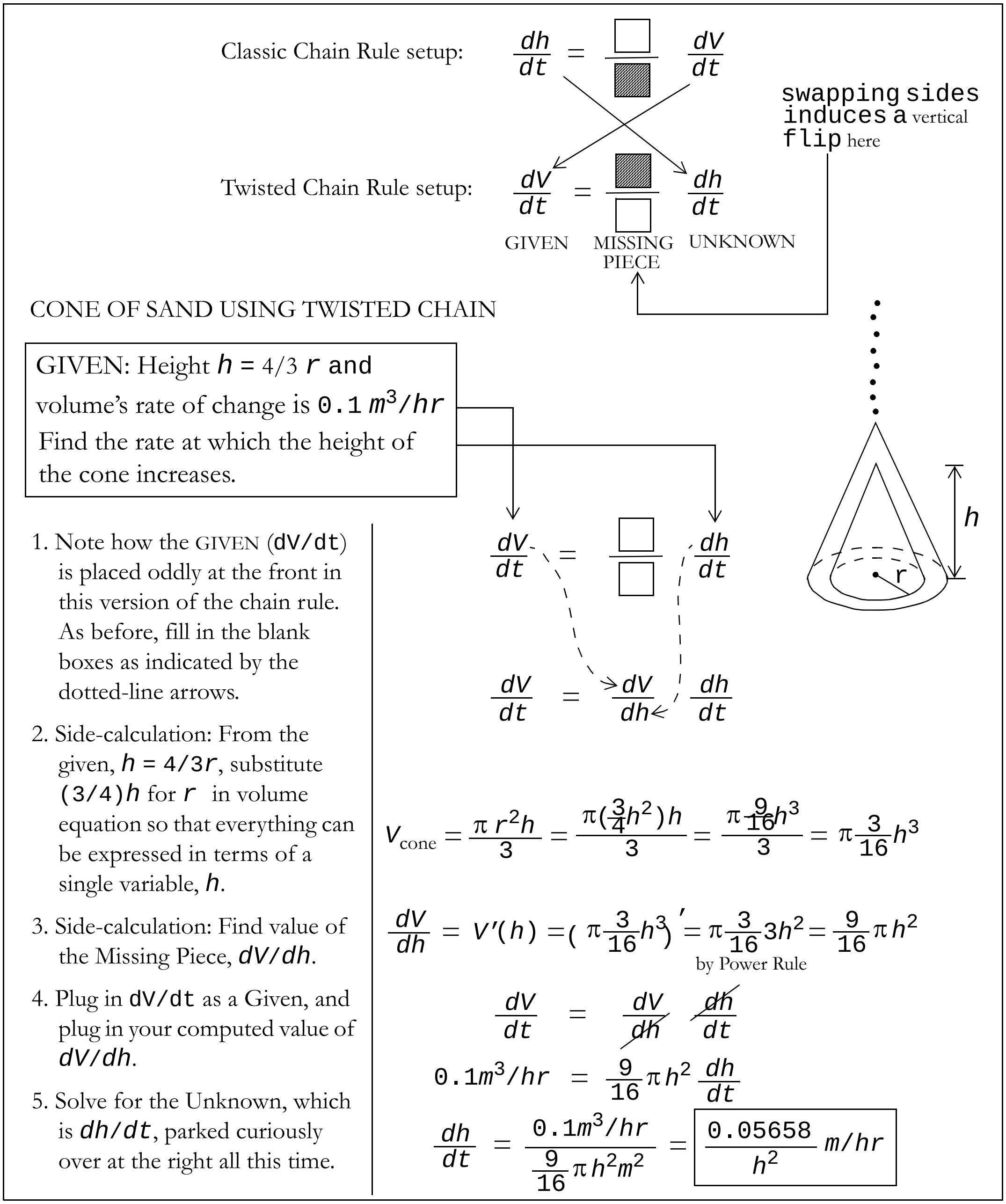


FIGURE 47: Twisted Chain Rule: Cone of Sand

One might wonder what dh/dt, the unknown, is doing parked way over at the right side of the equation during all but the final step shown in Figure 47. There are two ways to explain it. Looking at the Twisted Chain Rule naively, only in terms of its

mechanics, one can explain it as a devious (desperate) attempt to avoid the hairy kind of calculations shown next, in Figure **48**. Viewed this way, the Twisted Chain is just some fancy algebra. Alternatively, one can explain it from a theoretical standpoint, as ***implicit differentiation***, a technique whereby a variable gets pulled out of thin air and mysteriously tacked onto the right side of the equation.
(References: St. Andre, p. 149-150, Ryan, p. 128.)

CONE OF SAND USING CLASSIC CHAIN RULE

Something you probably *don't* want to do!

GIVEN: Height $h = 4/3\ r$ and volume's rate of change is $0.1\ m^3/hr$
Find the rate at which the height of the cone increases.

1. Fill in the blank boxes as indicated by the dotted-line arrows.

$$\frac{dh}{dt} = \frac{\square}{\square}\ \frac{dV}{dt}$$

UNKNOWN MISSING PIECE GIVEN

$$\frac{dh}{dt} = \frac{dh}{dV}\ \frac{dV}{dt}$$

2. Side-calculation: From the given, $h = 4/3r$, substitute $(3/4)h$ for r in volume equation so that everything can be expressed in terms of a single variable, h.

$$V_{\text{cone}} = \frac{\pi r^2 h}{3} = \frac{\pi(\frac{3}{4}h^2)h}{3} = \frac{\pi\frac{9}{16}h^3}{3} = \pi\frac{3}{16}h^3$$

3. Side-calculations: Solve the V_{cone} equation for h, then differentiate $h(V)$.

$$V_{\text{cone}} = \pi\frac{3}{16}h^3 \qquad h^3 = \frac{16V}{3\pi} = 1.69V$$

$$h = \sqrt[3]{1.69V} = 1.19V^{\frac{1}{3}}$$

$$\frac{dh}{dV} = \underset{\text{by Power Rule}}{h'(V)} = \frac{1.19V^{(\frac{1}{3}-1)}}{3} = 0.396V^{-2/3}$$

4. Set up the multiplication of the Missing Piece times the Given to find the Unknown.

$$\frac{dh}{dt} = \frac{dh}{dV}\ \frac{dV}{dt} = (0.396V^{-2/3})(0.1\ m^3/hr) = \frac{0.0396}{\sqrt[3]{V^2}}$$

5. Side-calculation to compute cube root of volume squared.

$$\sqrt[3]{V^2} = \sqrt[3]{\left(\pi\frac{3}{16}h^3\right)^2} = \sqrt[3]{9.869(.0351)h^6} = 0.7023h^2$$

6. Complete the calculation that was begun in step 4. (Same result as in Figure 47, but harder to obtain this way.)

$$\frac{dh}{dt} = \frac{0.0396}{0.7023h^2} = \boxed{\frac{0.05658}{h^2}\ m/hr}$$

FIGURE 48: Counterexample: Questionable Use of the Classic Chain Rule

To many people, taking the derivative of $h(V)$ instead of $V(h)$ will have a rather foreboding, inside-out quality about it, and in fact you wind up with negative fractional exponents and division by the cube root of something times the cone's height taken to the sixth power. Not a comfortable place to be, at least not when working on the clock, on an exam. So now you see the motivation for learning the Twisted Chain, which *somewhat* lightens the computational burden in cases similar to our Cone of Sand problem.

Partial Derivatives

See Partial Differentiation.

Partial Differentiation

"For two variables...there is a derivative in the x-direction and another in the y-direction and these may be obtained by a process similar to that for functions of one variable" (St. Andre, p. 143). There are several different notation schemes for denoting partial derivatives, only two of which are shown here, one using subscripts and one using the mirror-6 character to create a variation on the d/dx: theme:

$$f_x \qquad \frac{\partial f}{\partial x}$$

$$f_y \qquad \frac{\partial f}{\partial y}$$

$$f_z \qquad \frac{\partial f}{\partial z}$$

A partial derivative problem usually involves visualizing and drawing curves in three dimensions, but there are a few that can be worked out simplistically in a plug-and-chug manner. Here is one such problem which I chose because it brings into relief the essentials of holding two variables constant while manipulating the third:

Given: $f(x, y, z) = e^{xy} \ln z$
Find f_x, f_y, f_z

Holding y and z constant, differentiate f with respect to x:

$f_x = ye^{xy} \ln z$ (i.e., $x^0 \bullet y = 1 \bullet y = y$; $\ln z$ goes along for the ride as coefficient)

Holding x and z constant, differentiate f with respect to y:

$f_y = xe^{xy} \ln z$ (opposite case)

Holding x and y constant, differentiate f with respect to z:

$f_z = \frac{e^{xy}}{z}$ (i.e., the derivative of $\ln z$ is $1/z$; e^{xy} goes along for the ride as coefficient)

Note that the mirror-6 character is also used as a subscript indicating travel in the counterclockwise direction, e.g., in the context of a Green's Theorem integral. That has no connection with partial derivatives. Rather, it is a whimsied use based on the look of the character itself, because it suggests a counterclockwise swirl: ∂.

Partial Integration

Not defined here. For an example, see page **71** in **Chapter V: Integral Calculus**.

Part III: Integration Rules Continued (not as matched pairs)

Is gasoline or a spare tire more important when driving? In a sense, the spare tire is 'less important' — until you need it, then it is all important. Similarly, it is tempting to rank certain integration rules as more or less important. But by analogy with the spare tire, from one viewpoint they are 'all of equal importance'. Anyway, for what it's worth, after preparing a set of Calculus II flash cards for myself, I found that frequency of occurrence played out as follows:

49	Integration By Substitution (alias 'u-substitution' or 'w-substitution')
20	Integration By Parts (some with double LIATE)
17	Test & Tweak
16	Power Manipulation (including some FOILs followed by Power Manipulation)
13	Trig Identities
8	Complete the Square (and/or integrate $\int 1/(a^2 + w^2)$ directly)
7	Partial Fractions
4	Linear Substitution ($w = x+1$, $x = w-1$)
1	$\ln a\, a^x$ exponential (i.e., $\int a^x = a^x/\ln a$, reversed from $(a^x)' = \ln a\, a^x$)

These numbers were based in turn on our various homework assignments and quizzes. That's their 'empirical foundation', such as it is.

Integration By Substitution (alias 'u-substitution' or 'w-substitution')

First the *what*, later the *why*-it-works. (A student will want to know what the rule is and how to apply it, before wondering why it works, I assume. Accordingly, the *why* part I've deferred. In that section we relate Integration By Substitution back to the Chain Rule.)

Given: Integrate the following: $\int -3x^2 \sin x^3\, dx = ?$

1. Using a concept similar to the inner/outer notion that was introduced for the Chain Rule (page **83**), identify x^3 as the inner component of the expression under the integration sign. (The variable x^3 resides inside the function `sin( )`. That's what makes it the 'inner component'. Granted, this step takes some practice.)

2. Let $w = x^3$
   (Minor variant to note: In some books, e.g., in Salas & Hille, this whole business of Integration By Substitution goes under the name ***u-substitution***. In that

context, you would set $u = x^3$, not $w = x^3$. There is also a bit of 'culture' that goes along with the u- versus w-choice, but let's skip that.)

3. Take the derivative of w. Often this involves using the Power Rule, like:
$$dw/dx = (x^3)' = 3x^2$$

4. Solve for dx:
$$dx = dw/3x^2$$

5. Use w and $dw/3x^2$ to make substitutions in the integrand, as follows:
$$\int -3x^2 \sin(w)\, dw/3x^2$$
(Here, depending on your temperament, you might want to note in passing an odd asymmetry of the operation: x^3 we've merely replaced by a variable we like better, a new 'handle' if you will; dx, on the other hand, we have actually redefined as something else. Yet for both operations we use a single word: 'substitution'. That's a subtle lie.)

6. Strike out the two instances of $3x^2$ to cancel them:
$$\int -\cancel{3x^2} \sin(w) \frac{dw}{\cancel{3x^2}} = -\int \sin(w)\, dw$$
Also, move the minus sign 'through the integration sign' to its left:, treating –1 as a 'residual coefficient' that needs to be isolated from the upcoming integration.

7. Now you have a simple integration to perform, following one of the rules in Table **4** on page **81**:
$$-\int \sin(w)\, dw = -(-\cos w) + C = \cos w + C$$

8. Sub back x^3 since that is what w has stood for all this time:
$$\cos w + C = \cos(x^3) + C$$

9. If time permits, check your presumed answer by taking its derivative:
$[\cos(x^3)]' = -\sin x^3 (x^3)' = -\sin x^3 \bullet 3x^2 = -3x^2 \sin x^3$
We have come full circle to the integrand we were given at the outset. All is well. (What happened to '+C'? We ignored it, because it is always okay to be 'off by a constant'; see the **Constant Rule and '+C'** on page **78**.)

Now, *why* does all this work so magically (in problem after problem after problem, as it happens)? Recall that the Chain Rule (in its very first incarnation on page **83**) looked like this:

$$h(x) = f(g(x)) \qquad h'(x) = f'(g(x)) \bullet g'(x)$$

What Integration By Substitution does is reverse the work of the Chain Rule. As soon as we cleverly said "Let $w = x^3$" (at Step 2), we were in effect working our way back to $g(x)$ of the abstract chain shown above. Having thus 'recovered' $g(x)$, if you like, we were then able to take its derivative, which was found to be $3x^2$ (at Step 3). At that point, we had 'recovered' $g'(x)$ as well. Finally, all we needed was for *our* $3x^2$ to be in the denominator of a fraction so that it could cancel the *original* $3x^2$ in the given integrand. (This piece tends to be located on the left whereas $g'(x)$, its symbolic ancestor, is always on the right. That can be confusing.) The desired cancellation was made possible by a simple algebraic rearrangement of the pieces:

$$dx = \frac{dw}{3x^2}$$

Thus, the scary-looking $3x^2$ was made to vanish from the integrand, leaving us with a very manageable '$\sin(w)$' as the *only* thing to integrate. Hurray! Aren't you glad you asked?

The main thing to know: Integration By Substitution works like a charm, and it is recommended that you always try Integration By Substitution *first* before the half dozen other techniques. But how do you know it failed? You will know when you find yourself at the end of the calculations still unable to cancel the first term under the integral sign (i.e., still unable to cancel $-3x^2$, to put it in terms of the above example). Then it's time to abandon Integration By Substitution, and try the next most 'popular' technique: Integration By Parts, which is introduced next.

Integration By Parts

First the *what*, later the *why*-it-works. (A student will want to know what the rule is and how to apply it, before wondering why it works, I assume. Accordingly, the *why* part I've deferred. In that section we relate Integration By Parts back to the Product Rule, the one that it reverses.)

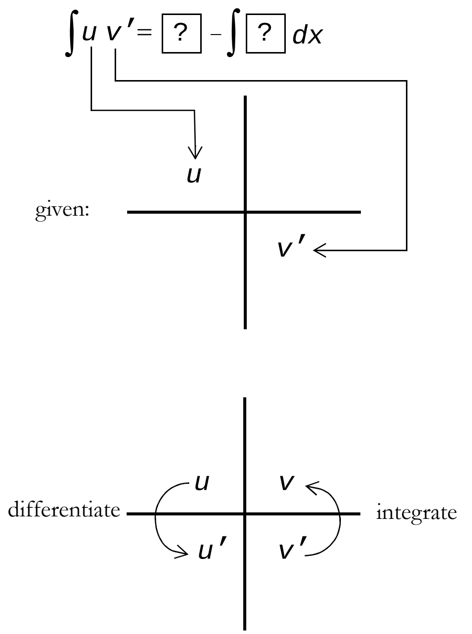
This is the formula for Integration By Parts:

$$\int u\,v' = u \bullet v - \int u'\,v$$

It looks pretty but what does it mean? It means you have been given a problem that contains an integral of form $u\,v'$ (or that's your theory, at least, for the moment), and you are hoping to trade in that difficult-looking integral for one whose form is $u'\,v$ instead (because *probably* the latter will be easier to do than the former; more about 'probably' when we get to the LIATE principle later). However, thus far the formula only tells us, "The integrand you don't want and the integrand you do want have such-and-such relation, with an equals sign and minus sign in-between." The question is how to actually populate the expressions $u \bullet v$ and $u'\,v$ so that you can carry out the integration and subtraction indicated on the right side of the equation. To help with this task of populating the variables (as I call it), there is a widely used trick, illustrated and annotated in Figure **49**.

The Generic Pattern

$$\int u\, v' = ?$$

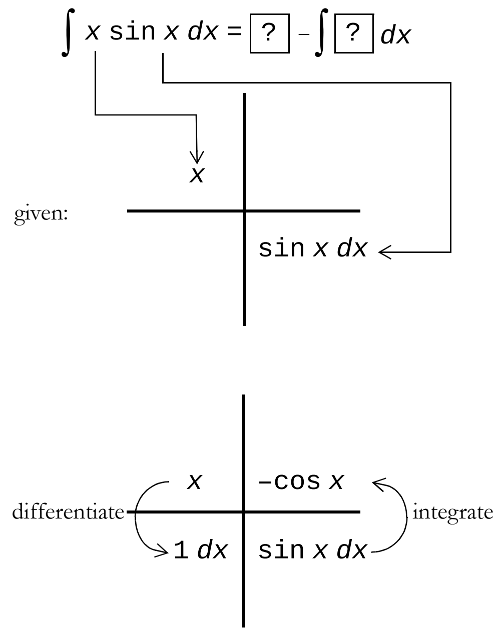


Now we have on hand all four ingredients that go into the formula for Integration By Parts:

$$\int u\, v' = u \bullet v - \int u'\, v$$

A Specific Example

$$\int x \sin x\, dx = ?$$



Having done a differentiation (by the Power Rule) and an integration (by a Trig Rule in reverse), as indicated immediately above, you now have on hand all the pieces you need to evaluate the right side of the equation:

$$\int x \sin x\, dx = x \bullet (-\cos x) - \int 1 \bullet (-\cos x)\, dx$$

$$= -x\,(\cos x) + \int \cos x\, dx$$

$$= \boxed{-x\,(\cos x) + \sin x + C}$$

Check by taking the derivative of the 'answer' (such as it is):

$$[\,-x\,(\cos x) + \sin x\,]' = x\,(\sin x) - \cos x + \cos x = x \sin x$$

by the Product Rule — by a Trig Rule

The point of it all? The integrand '1 • (–cos x)' we were *able* to handle. The original integrand, 'x sin x' we could *not* handle.

FIGURE 49: Workspace for Doing Integration By Parts — Generic and Specific

(Elsewhere you may see versions of the scheme presented in Figure **49** in which the four quadrants are employed differently, but all such variations on the theme accomplish the same thing: They provide a clean, open work space for organizing your thoughts, so that you can perhaps even enjoy your exam rather than feeling cornered and panicked by it!)

Having perused Figure **49**, you can say, "I understand what Integration By Parts is." Almost. From the example presented there, it *appears* that our procedure was simply to assign values to u and v' moving left to right through the given equation, letting x be u and letting $\sin x$ be v'. Actually, there's quite a bit more to the story. Enter the LIATE principle, as illustrated in Figure **50**. In Ryan, p. 193 (and elsewhere, e.g., wikipedia) the LIATE principle/trick/mnemonic/lifesaver (maybe literally, by having staved off untold calculus suicides?) is attributed to Herbert Kasube:

Logarithmic (example: $\int \ln x = ?$)

Inverse Trigonometric (example: $\int \arcsin x = ?$)

Algebraic (example: $\int x^3 - 1 = ?$)

Trigonometric (example: $\int \tan \theta = ?$)

Exponential (example: $\int 3^x - 1 = ?$)

From my little tribute above, I hope you get the idea that this is something Really Good. Here's the problem it solves: If we just blindly assign values to u and v' as we did in Figure **49**, we may or may not wind up with an easier integral to solve. By contrast, if one follows the LIATE hierarchy in assigning values to those two variables, one is guaranteed to discover a 'better' integral at the end of the process. In Figure **49**, I chose a problem that I knew would not contradict LIATE, even if we did it mindlessly, because I wanted to defer the discussion of LIATE. (It accords with LIATE because 'x' is algebraic and 'sin x' is trigonometric. Note how those two pieces of the given integrand accord with the LIATE sequence above.) For use in Figure **50**, I've chosen a problem where a mindless, left-to-right assignment of values would *not* have been good, and where the LIATE scheme tells us to assign to u and v' values the other way around. (This is represented by the broken arrows in Figure **50** that cross over one another.)

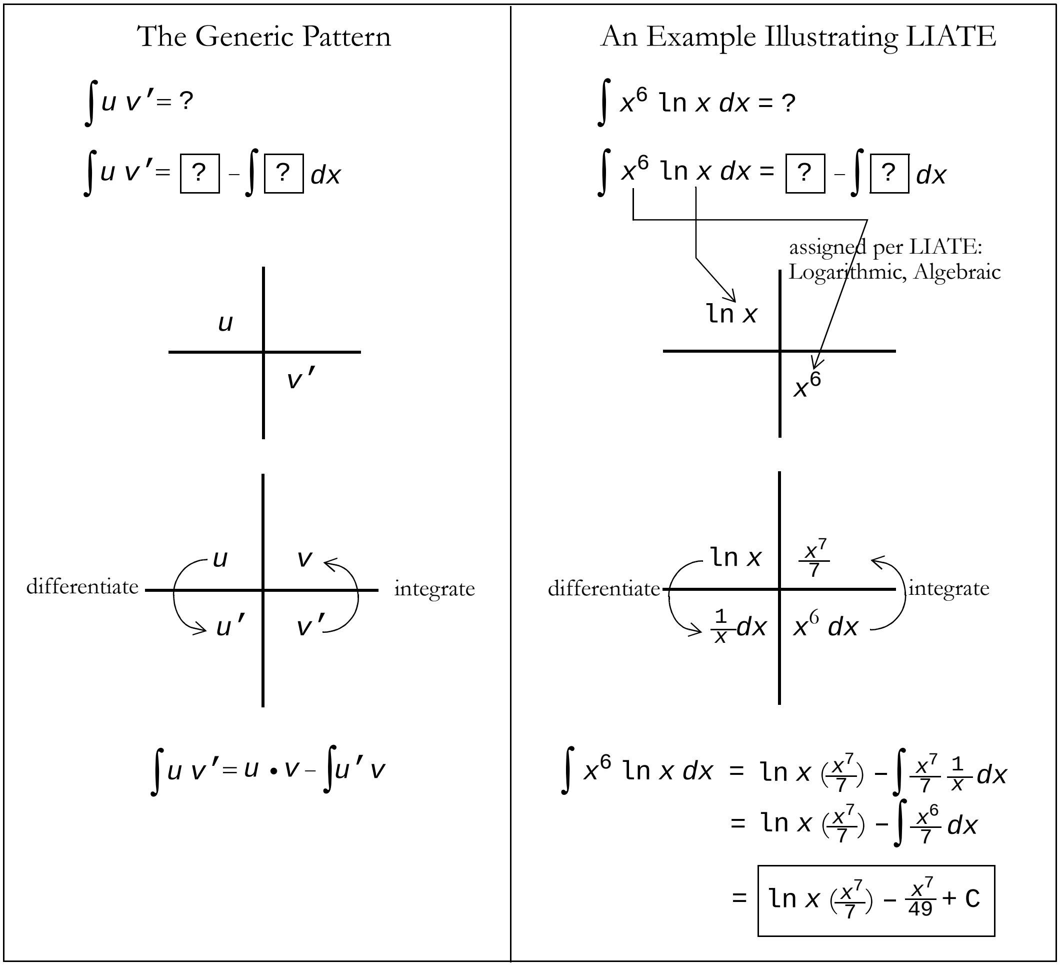


FIGURE 50: Example Illustrating the LIATE Principle/Trick/Mnemonic

In Figure **50**, the integrand contains an algebraic piece (x^6) and a logarithmic piece (ln x). Honoring the LIATE hierarchy, we assign ln x *first* (thus populating variable u in the scheme), and assign x^6 second (thus populating variable v'). This is the reverse of what we did in Figure **49**, where we were also compliant with LIATE but only by chance since we were not yet aware of LIATE. On the left side of Figure **50**, note that I've removed 'given' and the arrow this time since they imply a relationship that is too simple now that we are looking at a more realistic scenario. (For an even more realistic example that involves Integration By Parts, see **Modeling an Extremely Flat 'Wafer' of Urban Soot** on page **175**.)

Now for an explanation of why Integration By Parts works. Please refer to Figure **51**.

Recall that Integration By Parts reverses the Product Rule (page **83**), which looks like this (using the most informal style of notation for convenience):

$$(fg)' = f'g + fg'$$

We'll begin by putting all three parts of that equation under separate integral signs:

$$\int (fg)' = \int f'g + \int fg'$$

By exercising the cancellation property of the FTC, we can isolate (fg) on the left:

$$(fg) = \int f'g + \int fg'$$

Now if we simply solve for the expression on the far right, we will have placed the equation in the desired Integration By Parts form:

$$\int fg' = (fg) - \int f'g$$

$$\int uv' = u \bullet v - \int u'v$$

FIGURE 51: Derivation of Integration By Parts Formula

Note that Integration By Parts also covers reversal of the Quotient Rule, 'by inversion'.

(Variations on the theme: In some problems, you need to use both Integration By Substitution and Integration By Parts, together. In some problems, you may need to iterate Integration By Parts before a solution is found. In some problems, you may need to pretend there are 'parts', plural, even though there appears to be only one part. This is accomplished by conjuring an implicit '1' hidden just ahead of explicit 'dx' in the integrand, and integrating the '1' to 'x', thus populating the v' and v squares with 'something' even when it seemed that 'nothing' was there in the given problem to suggest any activity on that side of the grid. Strange though it sounds, this trick is perfectly legal because it simply reverses the following most rudimentary of all differentiation sequences: $x' = [x^1]' = x^0 = 1$. Note the close relation between this technique and the '+ C' trick for 'getting the constant back' by reversing $C' = 0$; page page **78**.)

Big picture: When taken together, Integration By Substitution and Integration By Parts account for 50% of the various problem types you are likely to encounter (as indicated by the rough-and-ready statistical profile near the beginning of this section). So, unless one of the seven other integration techniques immediately suggests itself to you as the key to the puzzle, you might as well begin your attack on a problem by testing out these two techniques, each in turn (or in combination).

Multiple Integrals/Double Integrals/Iterated Integrals

No rules here. For a discussion and example of iterated integrals, see page **71**.

Part IV: Other Rules

Rules for Limits, Continuity and Differentiability

See **Limits, Continuity, and Differentiability** on page **36**.

Not covered in this book: Rules for differential equations.

For an extended example that revolves around a differential equation, see **Dead Leaf Density** which starts on page **151**.

VII New Perspectives on Vector Calculus

Chapter Outline

Green's Theorem — in its Circulation Form AND Divergence Form

In the *Analects*, words of Confucius are recorded to this effect:

> If I hold up one corner and a student cannot come back to me with the other three, I do not repeat [his lesson]. 举一隅不以三隅反则不复也 < 论语 >

In other words, the cantankerous sounding Old Master is done with *that* student! Well, it's fine to clarify the student's share in the learning process, I suppose, but let's look at the flip side as well, the *teacher's* part of the contract.

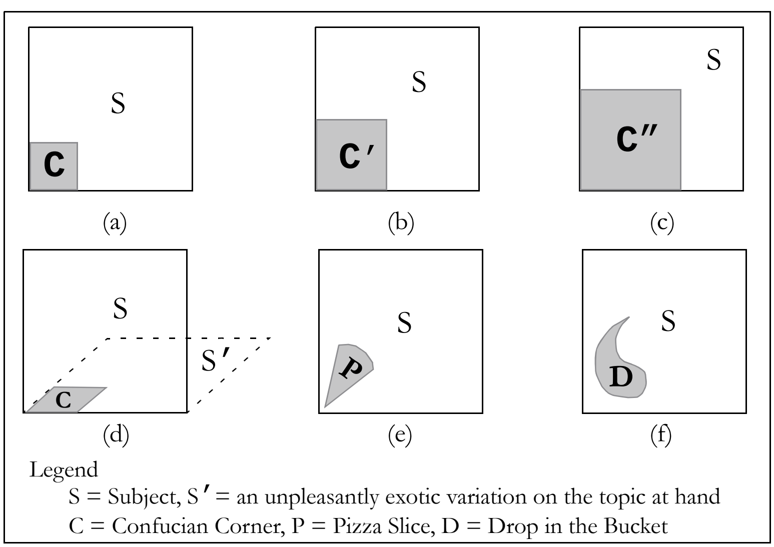


FIGURE 52: Some Legitimate & Perverse Variations on the Confucian Corner

It had darned well better be a *valid* corner of something that the master holds aloft. The teacher too has some obligations! I'm thinking of all the classes and textbooks where the student needs one of the valid corners shown as Figure **52**(a), (b), or (c) to be the curriculum content but is presented instead with one of the perverted 'corners' represented by diagrams (d), (e), and (f).

Figure **52**(d) depicts the case where a valid corner is presented, but the corner seems to belong to something *other* than S, apparently the author's pet related subject, which we designate by S' (using prime notation in its ordinary role, nothing to do with derivatives for the moment). Example: From its title, Bressoud's *Second Year Calculus* might seem to be a resource for students of our 'vintage calculus', as defined on page **3** above, as they embark upon a Calculus III course. But it is no such thing. As it happens, the phrase 'second-year calculus' has an entirely different meaning to those in the know, i.e., the serious math students. For them, 'second-year' is a code word for, 'Here is where you depart from vintage calculus and receive your initiation into wonk calculus'. For more about this, see notes **2** and **3** on page **229**.

Figure **52**(e) depicts a variation on the theme: The 'Pizza Slice' curriculum which is undeniably *of* the subject, S, but is so oddly shaped that it cannot possibly lead a student *to* the whole square, never mind how large or small a slice it might be.

Figure **52**(f) is similar, depicting instead the proverbial 'Drop in the Bucket'. Again, no matter how large or small the 'drop' may be, it cannot possibly show the way out to the four borders of S, which is the student's reasonably *presumed* framework of the curriculum.

The problem makes itself felt on both the macro-scale and micro-scale (which is to say, fractally, if you like). At the macro-scale, the defect will be found in the general outline of a calculus course or book about calculus. This book, for instance, suffers from several of the (d)(e)(f)-type maladies, I'm sure. But now at the micro-scale, my aim is to make it up to the reader regarding Green's Theorem at least.

Among the many presentations I've seen, only Wood's (see **Bibliography**) strikes me as a good-faith effort to provide a well-formed 'Confucian corner' from which the student can reasonably be expected to extrapolate the whole of Green's Theorem. If Wood's presentation is Figure **52**(b), then ensuing pages are my attempt to take it, by way of some visual embellishments, toward Figure **52**(c), let's say.

A few general remarks about the place of Green's Theorem in the larger scheme of things: In some presentations, Green's Theorem is the focal point, and it even spins off two hybrid types (e.g., in Stewart, p. 1114) that I refer to below as Green-toward-Stokes and Green-toward-Gauss. Nahin is one who takes Green as the focal point, using the name 'Green's Theorem' as a *generic* term to encompass the whole trifecta of theorems associated with the names of Green, Stokes and Gauss (Nahin, p. 204-208). Approaching it the opposite way, Schey seems at pains to avoid even mentioning Green once, save in a problem that appears at the bottom of page 148. That seems odd, but it is not so different from Spivak's approach. Spivak postpones talking about Green's Theorem until he can slip it in almost parenthetically as 'a very special case of Theorem 5-5' which is to say a very special case of Stokes. For Spivak, 'Stokes' Theorem' would be the all-encompassing generic term. See Spivak, pp. viii, 124 and 134; compare discussion of Figure **63** below. In a confused echo of the others, Bressoud too treats the three theorems as essentially one, but he tries using the Divergence Theorem (aka Gauss's Theorem) as his focal point, on p. 297-305, with strange results, more like an encrypted message than an expository passage for humans to read. (Nomenclature note: Nowadays, Gauss's Theorem is commonly referred to as The Divergence Theorem; I follow Salas & Hille in preserving the older name.)

For all that variety in how to approach 'Green's Theorem' (with overtones of historical controversy and 'politics'), the *main* issue regarding it is that there is not just *one* of it, even down at the supposed root of the tree of related theorems. Rather, there exists inherently, at the very bottom of it all, a perfectly balanced *pair* of theorems that together comprise 'Green's Theorem'. This point is addressed tacitly and fixed perfectly in Wood's presentation which reveals the whole picture for once.

Please refer next to Figure **53** where I summarize some of the nomenclature issues that surround Green's Theorem. The presentation proper begins with Figure **54**, which is modeled on Wood, pp. 2-3.[22]

Reminder: This is not a textbook, although parts of it may seem to adopt a very textbook-like approach (e.g., Chapters **I** and **VI**, and **Appendix C**). For this final topic, vector calculus, I've opted to jump into the deep end of the pool, with many definitions skipped over (e.g., What is an 'outward unit normal vector'? Why are the letters **i**, **j** and **k** bolded?) Many of those details must be supplied from elsewhere by the reader. So in that sense I really am offering only 'one corner'. However, since my 'one corner' is based closely on Wood's solid presentation, and since my embellishments to her presentation are highly visual, I think much can be gleaned (in this part and ensuing parts of the chapter) without the full technical preparation.

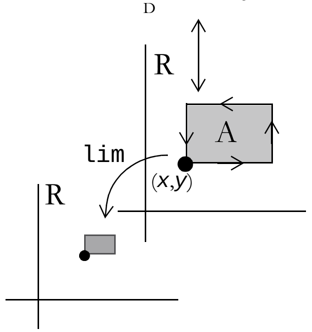
Green's Theorem

Canonical Form: $$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy$$

Pragmatic Form: $$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

circulation version (curl)

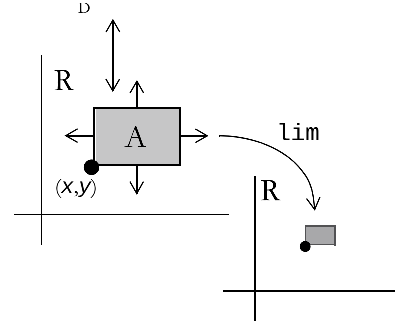
$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$



For detailed derivation of curl in Green (after Wood), see page 108.

divergence version (flux alias div)

$$\int_C P\,dy - Q\,dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA$$



For detailed derivation of flux (aka div) in Green (after Wood), see page 111

Green-toward-Stokes

$$\oint_C \mathbf{F}\bullet d\mathbf{r} = \iint_D \operatorname{curl}\mathbf{F}\bullet\mathbf{k}\,dA$$

Stokes' Theorem

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$$

Green-toward-Gauss

$$\oint_C \mathbf{F}\bullet\mathbf{n}\,ds = \iint_D \operatorname{div}\mathbf{F}(x,y)\,dA$$

Gauss's Theorem

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\cdot dV$$

(for continuity, I show all four of these in FTC Pragmatic Form)

The subtext: This form aligns it with *the* FTC. This is the Canonical Form. (Regarding FTC Canonical Form and FTC Pragmatic Form, please refer to page 210.)

The subtext: We've swapped sides to achieve 'Hard Explained By Easier' order; which is to say FTC Pragmatic Form.

Here, *if* we are lucky enough to have seen Wood's presentation (or another like it), we learn that the formula presented in textbooks as 'Green's Theorem', whether in Canonical Form or in Pragmatic Form, is in any event only *one of two* mirrored versions of Green's Theorem, one for circulation (curl), another for divergence (flux).

The confused treatment of Green's *curl* vs. Green's *flux* in conventional presentations is all the more aggravating because of the following circumstance: Stokes' Theorem is associated exclusively with *curl*, and Gauss's Theorem (alias the Divergence Theorem) is associated exclusively with *flux*. Meanwhile, Green's Theorem *seems* to be associated exclusively with *curl*, as conventionally presented. But really it anticipates both Stokes and Gauss. Nevertheless, textbook authors follow the old precedent of speaking arbitrarily of Green's *curl* version as *the* Green's Theorem (followed by a hodgepodge of qualifying statements and amendments in a belated attempt to backfill the complete story). Thus, an essentially simple relation is turned quite needlessly into an astonishing mess (in the student's mind), a mess that can only be compounded by the business about Canonical Form getting silently flipped around to Pragmatic Form, another red herring that obscures the inherent simplicity of the underlying patterns.

FIGURE 53: Green's Theorem and Beyond

In Figure 54, we expand on the first of the rectangular icons seen in Figure 53 , the one on the *curl* side of the diagram. Following Wood, I build the rectangle on an anchor point (x, y), which becomes the focal point for taking the limit later, as depicted schematically on the right side of Figures 54-58.

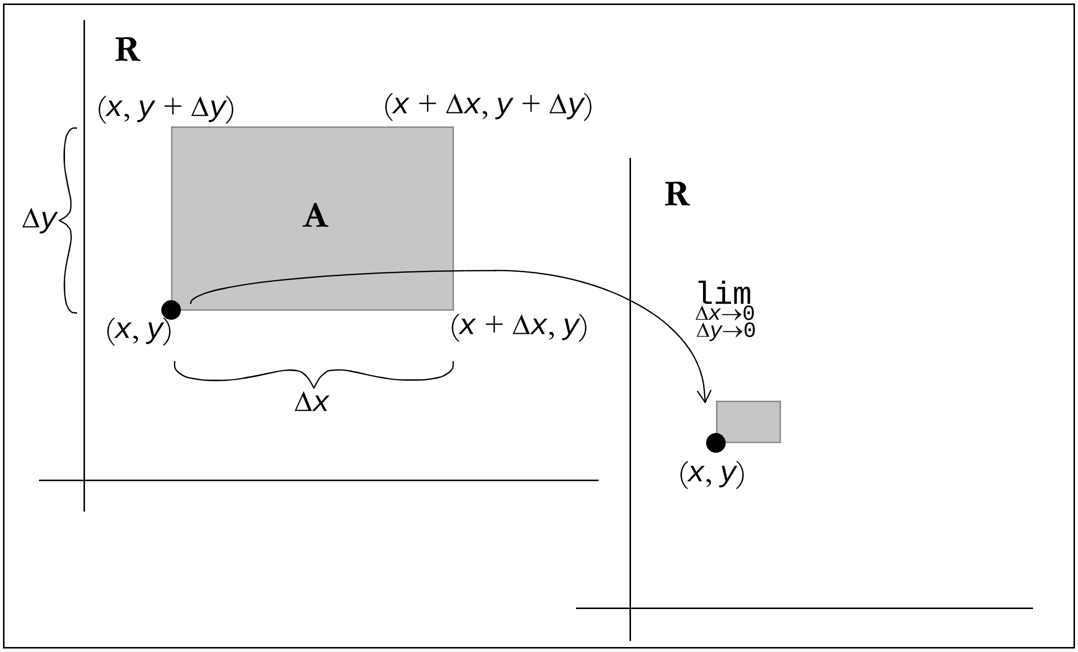


FIGURE 54: Special Coordinates for Rectangular Area ‘A’ in Region ‘R’

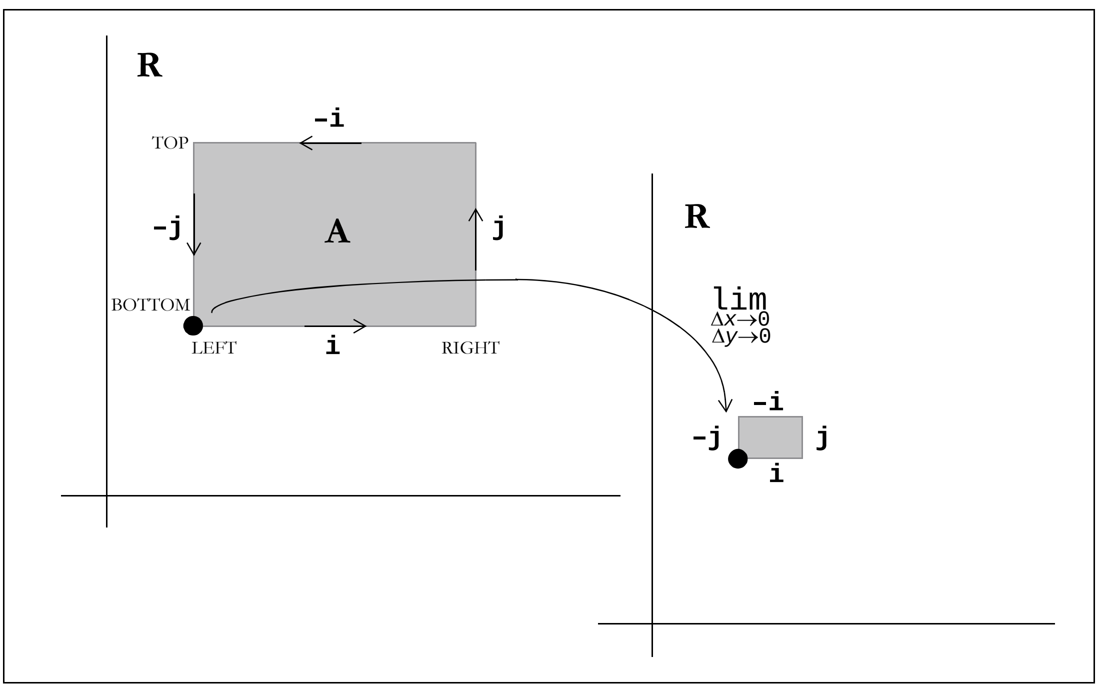


FIGURE 55: Area A Repeated, now with *curl* Vector Labeling

Figure 55 presents the same object in the same region but with different labels, now emphasizing its vector aspect.

Curl Derivation (circulation density at a point)

Bottom (**i**)	$\mathbf{F}(x, y) \bullet \mathbf{i}\, \Delta x$	$= P(x, y)\, \Delta x$
Top (–**i**)	$\mathbf{F}(x, y + \Delta y) \bullet (-\mathbf{i})\, \Delta x$	$= -P(x, y + \Delta y)\, \Delta x$
Right (**j**)	$\mathbf{F}(x + \Delta x, y) \bullet \mathbf{j}\, \Delta y$	$= Q(x + \Delta x, y)\, \Delta y$
Left (–**j**)	$\mathbf{F}(x, y) \bullet (-\mathbf{j})\, \Delta y$	$= -Q(x, y)\, \Delta y$
Top + Bottom	$-[P(x, y + \Delta y) - P(x, y)]\, \Delta x$	$\approx -\left(\frac{\partial P}{\partial y}\Delta y\right)\Delta x$
Right + Left	$[Q(x + \Delta x, y) - Q(x, y)]\, \Delta y$	$\approx \left(\frac{\partial Q}{\partial x}\Delta x\right)\Delta y$

Gradually, as you work through this derivation, it will become apparent why the coordinates for rectangular area A are written as they are, exclusively in generic terms, with symbols, rather than with familiar cartesian value pairs such as (1,1) or (3,1). In the definitions of the four sides of the rectangle as calculated above, note how the focus of attention is always the bottom left corner. The other thing happening in the first four rows of calculation is that vector forms are being translated to scalar forms for each side of the rectangle, in turn. Now that we've obtained the two expressions that use the mirror-6 character (∂), the next step is to sum them algebraically, with $\Delta x \Delta y$ factored out:

$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\Delta x \Delta y$$

As indicated in Figure 56, what this amounts to is a special way of expressing the perimeter of the rectangle in question.

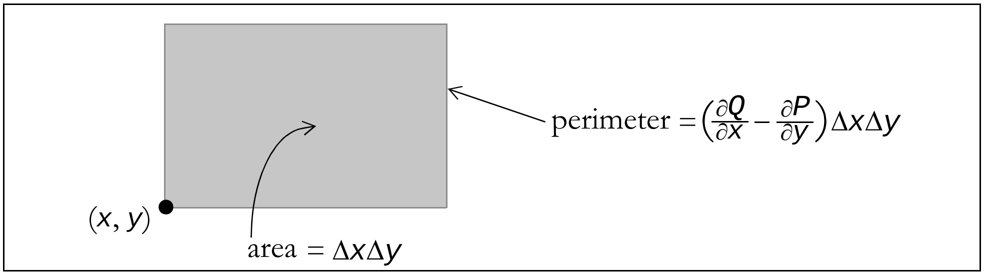


FIGURE 56: Area and Perimeter

Already, this unusual way of labeling a rectangle provides a premonition of what will follow. Taken together, the two formulas shown in Figure 56 say, in effect, that the expression in parentheses is a key to the relation between the perimeter and area of any rectangle. If we divide one by the other, thus cancelling the two $\Delta x \Delta y$ terms, what remains? An estimate of 'density of circulation' for the rectangle. Or, instead of doing the division, suppose we take the limit as $\Delta x \to 0$ and $\Delta y \to 0$. The rectangle shrinks to its anchor point, (x, y), as depicted impressionistically on the right side of Figure 54. We are left (again) with...

$$\boxed{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}$$

...now representing the *circulation density* alias *curl* of a vector $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ at the point (x, y).

The boxed expression you will recognize from the top of Figure 53 where it appears as the *derivative* under the double integral sign. (I.e., its busy notation notwithstanding, it occupies exactly the slot in Green's Theorem that $f(x)$ occupies in the FTC.) When expressed using the synonymous term *curl* **F**, it provides a preview of or bridge to Stokes' Theorem as well.

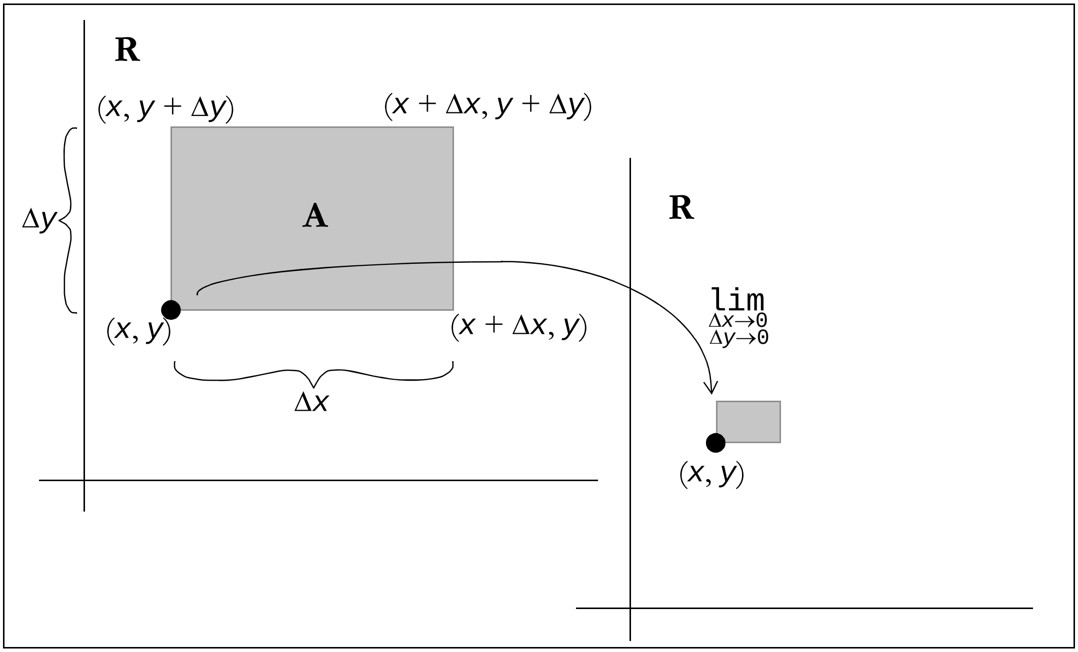


FIGURE 57: Special Coordinates for Rectangular Area ‘A’ (repeated)

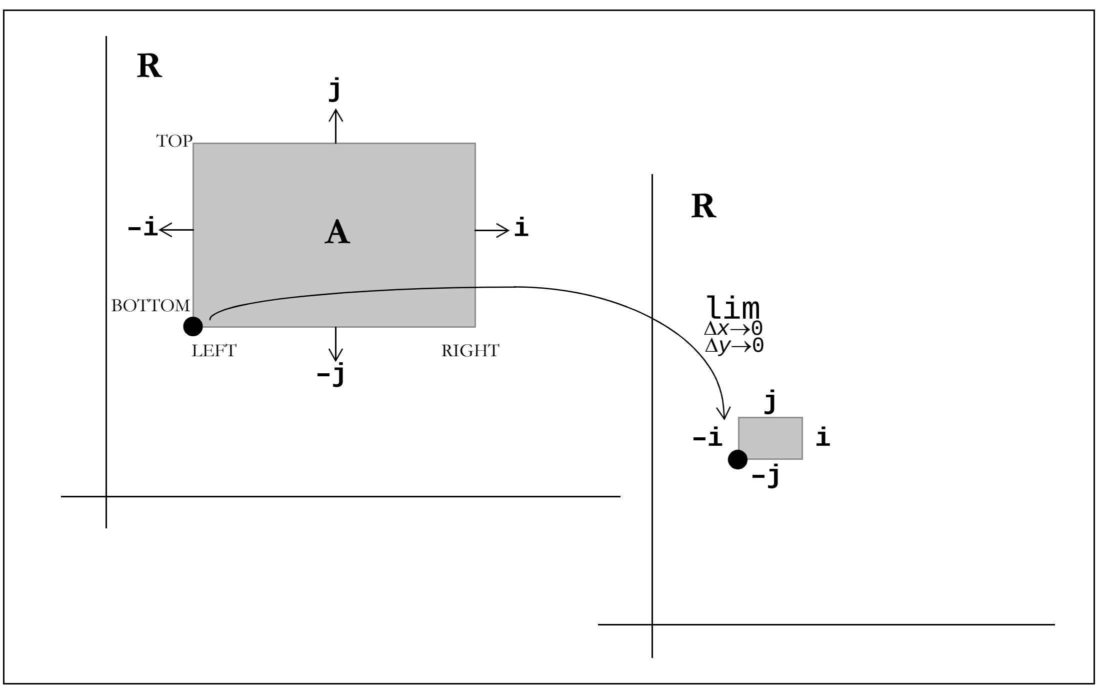


FIGURE 58: Area A Repeated, now with *div* Vector Labeling

Divergence (div) Derivation (flux density at a point)

Top ($\mathbf{j}$)	$\mathbf{F}(x, y + \Delta y) \bullet \mathbf{j}\,\Delta x$	$= Q(x, y + \Delta y)\,\Delta x$
Bottom ($-\mathbf{j}$)	$\mathbf{F}(x, y) \bullet (-\mathbf{j})\,\Delta x$	$= -Q(x, y)\,\Delta x$
Right ($\mathbf{i}$)	$\mathbf{F}(x + \Delta x, y) \bullet \mathbf{i}\,\Delta y$	$= P(x + \Delta x, y)\,\Delta y$
Left ($-\mathbf{i}$)	$\mathbf{F}(x, y) \bullet (-\mathbf{i})\,\Delta y$	$= -P(x, y)\,\Delta y$
Top + Bottom	$[Q(x, y + \Delta y) - Q(x, y)]\,\Delta x$	$\approx \left(\frac{\partial Q}{\partial y}\Delta y\right)\Delta x$
Right + Left	$[P(x + \Delta x, y) - P(x, y)]\,\Delta y$	$\approx \left(\frac{\partial P}{\partial x}\Delta x\right)\Delta y$

Again, in the first four rows of calculation, vector forms are being translated to scalar forms. As before, we sum the final two expressions, with $\Delta x\Delta y$ factored out:

$$\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\Delta x\Delta y$$

Now take the limit as $\Delta x \to 0$ and $\Delta y \to 0$. The rectangle with area A shrinks to its anchor point, (x, y), depicted impressionistically on the right side of Figure **57**. We are left with...

$$\boxed{\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}}$$

...as the *flux density* alias *divergence* of a vector $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ at the point (x, y).

The boxed expression is the derivative under the double integral sign in the *other* 'Green's Theorem' (down the right-hand path in Figure **53**) — the one mentioned belatedly or never, as such, in many presentations. When expressed using the synonymous term *div* **F**, it provides a preview of or bridge to Gauss's Theorem (aka the Divergence Theorem) as well.

From the discussion above, you can now see another reason I was thinking about the 'Confucian corner'. In the rectangle with anchor point, we find a rough analog to it, albeit working in reverse by way of a limit process. But in 'taking the limit' have I gone against my own admonition (in **Chapter II**) not to perform a flea-hop? No, because the process involves an 'imposed limit' not an 'inherent limit', as described in **Appendix D**. The process does not conclude by asserting that the rectangle or its anchor point is not there — has been annihilated into the black hole of a zero-dimensional point. The process only says we wish to shrink the rectangle down *to* its anchor point, thus making its area negligible. This is the legitimate way of using the limit concept.

GREEN'S THEOREM — CIRCULATION alias CURL alias TANGENT VERSION

$$\oint_C \mathbf{F} \bullet \mathbf{T}\, ds = \oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

The counterclockwise circulation of a field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ around a simple closed curve C in the plane equals the double integral of the curl F [a higher dimensional analog to a derivative] over the region D enclosed by C.

A direct, 2D representation of curl, using tangents (T) of varying length:

An indirect representation, employing a 'picket fence' that stands up in the third dimension, to visually enhance the data's representation:

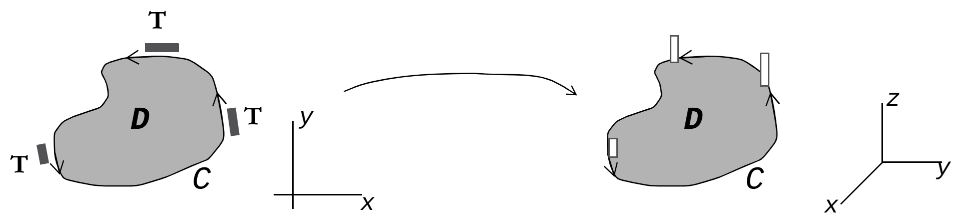


FIGURE 59: Green's Theorem — Circulation Version

GREEN'S THEOREM — DIVERGENCE alias FLUX alias NORMAL VERSION

$$\oint_C \mathbf{F} \bullet \mathbf{n}\, ds = \oint_C P\,dy - Q\,dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA$$

The outward flux of a field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ across a simple closed curve C in the plane equals the double integral of div F [a higher dimensional analog to a derivative] over the region D enclosed by C.

A direct, 2D representation of the divergence, following the outward unit normal vectors (**n**):

An indirect representation, employing a 'picket fence' that stands up in the third dimension, to visually enhance the data's representation:

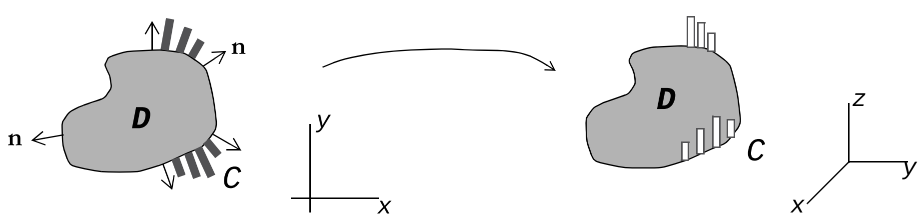


FIGURE 60: Green's Theorem — Divergence Version

This concludes our tour of Green's Theorem(s), which followed for the most part Beth Wood's presentation. Her approach is valuable because it shows, up front, both the circulation version and divergence version *of* Green's Theorem.

The pictures that I've interleaved as Figures **59** and **60** above are based on an illustration of the Line Integral using a " 'fence' or 'curtain' " in Stewart, p. 1082. Such images may seem fanciful at first, as though new dimensions are to be conjured up like genies in Calculus III. But the motivation is practical. And if you think about it, a very similar concept must lie behind the custom (nowadays taken for granted) of showing area under a curve in Calculus I/II: The xy-points that define a curve in that context *could* be plotted solely along the x-axis, by the logic that the FTC is 'for Scalar Functions on One Variable' (Bressoud, p. 279). But this would create a nearly useless hodgepodge of overlapping data. For example, the parabola sketched on page **20** might take on this appearance, as one endeavored to be a 'purist' and show everything on a single, heavily annotated axis:

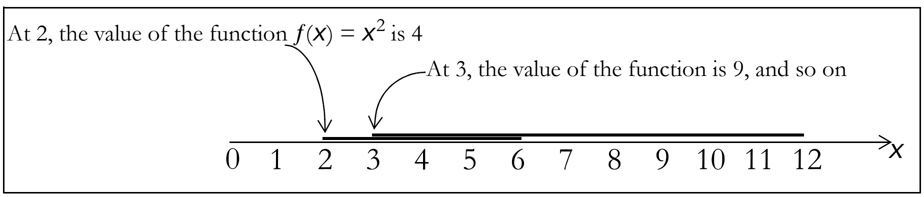


FIGURE 61: Parabola Plotted on the x-axis Only

In Figure **61**, not only is the shape of the parabola sacrificed, but even the individual data points such as (2,4) and (3,9) become obscure. To prevent such clutter, the data is normally made to 'stand up' in the y-direction, not unlike the notional 'picket fence' erected *on* a planar curve when modeling a Line Integral or Green's Theorem. And that is what creates, out of thin air, the area *under* a curve in Calculus I/II. For either of these operations, in the instant before someone carried it out for the very first time it must have seemed fanciful or 'extradimensional', but immediately afterwards both would have reason to appear physical and real, which they are. (This thread is resumed in **Implicit y, Implicit z, Implicit w** below.)

An example of the circulation version of Green's Theorem follows.[23]

Green's Theorem Example, Circulation Version, Both Sides of the Formula

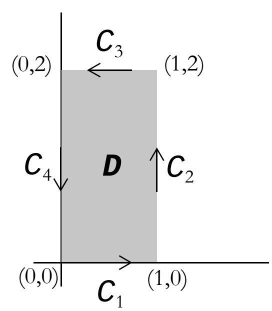
Prologue. The circulation version of Green's Theorem, written in FTC Pragmatic Form, has this familiar look:

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Usually, the left side represents the problem to be solved and the right side is the part you evaluate to discover the answer. In the ensuing example, we will evaluate both sides by way of demonstrating their respective natures, including the contrast wherein the line integral (left side) is relatively hard to tackle, and the double integral (right side) easier. On an exam, this routine of evaluating both sides, each in turn, is often referred to as 'validating Green's Theorem' or 'validating Stokes' Theorem' (see page **143** for an example of the latter). (An aside: A more realistic description of the procedure would be 'validating *my work*' since 150 years of scrutiny can hardly leave much doubt about the theorem itself, only a concern in the student's mind that she may not succeed in producing the same answer twice for both parts of the problem!)

Problem. Find the counterclockwise circulation of the field $\mathbf{F} = -xy\,\mathbf{i} + x^2\,\mathbf{j}$ around the boundary of the rectangle (0,0), (1,0), (1,2), (0,2). Expressed in scalar terms, and with the rectangle sketched to its side, the problem to solve is this:

$$\oint_C -xy\,dx + x^2\,dy$$



Part 1: Evaluate the line integral directly (i.e., work with the left side of Green's Theorem as written in FTC Pragmatic Form).

The rectangle falls naturally into four segments that may be parameterized and treated as four separate line integrals as follows:

C_1: $x = t$, $dx = t' = 1\,dt = dt$, $y = 0$, $dy = 0\,dt$, $0 \le t \le 1$

C_2: $x = 1$, $dx = 0$, $y = t$, $dy = dt$, $0 \le t \le 2$

C_3: $x = 1-t$, $dx = (1-t)' = (0-1)dt = -dt$, $y = 2$, $dy = 0\,dt$, $0 \le t \le 1$

C_4: $x = 0$, $dx = 0\,dt$, $y = 2-t$, $dy = (2-t)' = (0-1)dt = -dt$, $0 \le t \le 2$

Plugging these new definitions for x, y, dx and dy into the function $-xy\,dx + x^2\,dy$ and integrating each of the four segments in turn, we have:

$$\oint_{C_1} -xy\,dx + x^2\,dy = \int_0^1 -t\ (0)\ dt + t^2(0\ dt) = 0$$

$$\oint_{C_2} -xy\,dx + x^2\,dy = \int_0^2 -t\ (0)\ dt + 1^2(dt) = \int_0^2 1\ dt = \Big|t^1\Big|_0^2 = 2$$

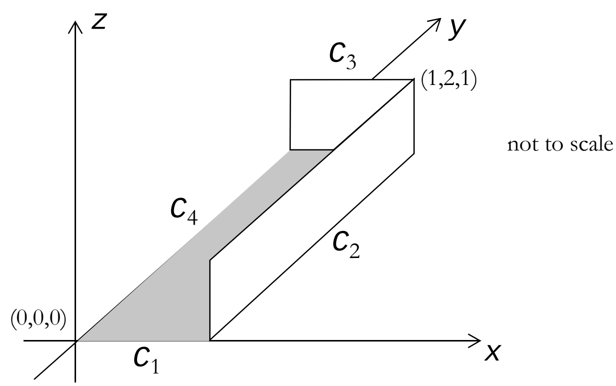
$$\oint_{C_3} -xy\,dx + x^2\,dy = \int_0^1 -(1-t)(2)(-dt) + (1-t)^2(0\ dt) = \Big|2t - \tfrac{2}{2}t^2\Big|_0^1 = 2 - 1 = 1$$

$$\oint_{C_4} -xy\,dx + x^2\,dy = \int_0^2 -0(2-t)(0\ dt) + 0^2(-dt) = 0$$

Now arithmetic is used to sum the four intermediate steps:

$$\oint_{C} -xy\,dx + x^2\,dy \quad = \oint_{C_1+C_2+C_3+C_4} -xy\,dx + x^2\,dy \quad = \quad 0 + 2 + 1 + 0 \ = \boxed{3}$$

If we add a z-axis to the original diagram, the two non-zero integrals can be made visible as a kind of ‘fence’ that runs along the C_2 and C_3 segments of the perimeter:



Part 2: Evaluate the double integral over region D to find the line integral. (In other words, only now we are actually using Green’s Theorem, to solve the problem *in*directly. *This* is the theorem’s normal use.)

$$\oint_{C} P\,dx + Q\,dy = \iint_D \Big(\tfrac{\partial Q}{\partial x} - \tfrac{\partial P}{\partial y}\Big) dA = \iint_D \Big(\tfrac{\partial}{\partial x}x^2 - \tfrac{\partial}{\partial y}(-xy)\Big) dA$$

$$= \int_0^1 \int_0^2 \ 2x + x\ dy\ dx$$

$$\int_0^2 2x + x\ dy \ = \ \Big|2xy + xy\Big|_0^2 \ = \ 4x + 2x$$

$$= \int_0^1 4x + 2x\ dx \ = \int_0^1 6x\ dx \ = \ \Big|\tfrac{6}{2}x^2\Big|_0^1 \ = 3 \bullet 1 = \boxed{3}$$

Besides illustrating Green's Theorem (in its circulation version), the above example serves to demonstrate the contrast between the 'hard' and 'easier' side of a Calculus III formula. (To the neophyte, Part 1 and Part 2 above might look equally daunting. Here you'll have to trust me: With practice, the kind of calculation shown in Part 2 becomes a pleasant game whereas the parametric equations [page **221**] required in Part 1 remain [for some of us at least!] forever challenging. Note also that the problem I devised above is close to being the *minimal* Green's Theorem problem, just for illustrative purposes...)

The hard/easier contrast leads in turn to the unwritten law of H ∃ E (page **122**) and thence to the silent and hitherto nameless practice of writing certain Calculus III formulas in FTC Pragmatic Form (page **210**) in preference to FTC Canonical Form (a deplorable practice in my opinion).

More about Green's Theorem:

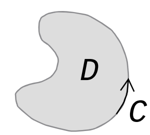
- A derivation of the version I called Green-toward-Stokes (i.e., the vector form of the circulation version of Green's Theorem) is discussed on page **136**.
- Figure **65** on page **138** is related to the vector form of the divergence version.
- For a physical description of curl and flux, see Schey pp. 86-91 and 31-33, respectively.

Toward a Unified Geometric Profile of the Calculus III Landscape

This section of the chapter will culminate in **Table 7** and Figure **63**, where I attempt a distillation of a dozen FTC variants to a high level of abstraction. In doing so, my aim is to provide the Calculus III student with a bird's-eye view of the whole terrain. That's the hoped for virtue of the scheme, but with any such abstraction comes the danger of drifting too far away from reality. Phrased in military terms, the admonition would be that 'the map is not the territory'. Accordingly, by way of establishing a few patches of 'territory' before we move up into the thin air of the 'map' (comprised of **Table 7** and Figure **63**), I will paraphrase or quote from Protter and Morrey, where one finds especially good definitions of the three primary theorems of vector calculus, definitions that manage to be at once technical *and* reader-friendly:

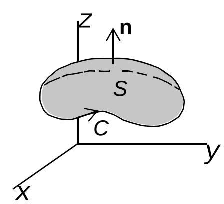
GREEN'S THEOREM

> Given: **v** is a vector function over a closed curve *C*. Green's Theorem is a formula that connects the double integral of the *derivative* of **v** taken over *D* with the line integral of **v** *itself* over *C*.
> — after Protter & Morrey, p. 445, with order switched to FTC Canonical Form, also with labels changed, sketch added, and italics added



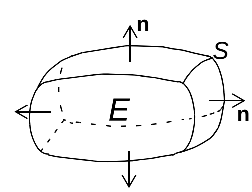
STOKES' THEOREM

> The theorem of Stokes establishes an equality between the integral of curl **v** • **n** [a higher dimensional analog to the *derivative*] over a surface S and the integral of **v** [*itself*] over the boundary of S.
> — Protter & Morrey, p. 477, sketch added



GAUSS'S THEOREM (aka the Divergence Theorem)

> ...determines the relationship between an integral of the *derivative* of a function over a three-dimensional region in $\mathbf{R}^3$ [i.e., a volume] and the integral of the function *itself* over the boundary of that region.
> — Protter & Morrey, p. 486, italics and sketch added



Note the occurrence of '*derivative*' and '*itself*' in all three definitions, as tweaked by me, with italics added to prevent these two key terms from getting lost in the shuffle. (By the way, in wonk calculus the letter *d* is cast in a high-profile role which *guarantees* visibility at all times for this pattern that keeps threatening to fade into the background of the thorny notations of vintage calculus.[24])

Degrees of abstraction: Stokes' Theorem as defined immediately above seems relatively *concrete*, even chatty or verbose if compared to its representation in Figure **63** as a sphinx-like icon. Conversely, if compared to the statement of Green's Theorem given on page **113** (after Wood, pp. 3-4), the synopsis of Green immediately above looks *abstract* (even abstract to the point of being deficient, since it follows the unfortunate group-think tradition of letting the curl version of Green serve arbitrarily as the stand-in for both versions). In lieu of the third definition given above, one might have said, "Gauss's Theorem relates a volume integral to a surface integral" (after Schey, p. 45). Would that have been a 'helpful synopsis written in friendlier language' or would it have been 'too abstract to mean anything useful'? Again, your perspective or bias will vary depending on where you are in the labyrinth at a given moment.

Another preliminary step: Before classifying the FTC variants and related formulas according to their geometric identities, we need to build a suitable list of candidates that will populate the 'landscape'. That is the purpose of **Table 6**. There I present an expanded version of the vector calculus summary table found in Stewart, p. 1152, with his graphics temporarily stripped out and with his notation slightly modified.[25] The five FTC variants that appear left-justified are the ones that match Stewart's list. To suggest a more complete picture of the Calculus III landscape, I've added eight formulas to his five. These extra eight are right-justified in **Table 6** to help keep them segregated from his more fundamental list of five. Several of these extra eight are not FTC variants per se, only FTC-*related* formulas that play prominent roles in Calculus III.

TABLE 6: FTC Variants and Related Formulas

Name of FTC Variant	Five FTC Variants after Stewart p. 1152 (reordered) Interleaved With Eight Related Formulas (right-justified)
Leibniz (*the* FTC)	$\int_a^b f(x)\,dx = F(b) - F(a)$
Iterated Integral	$\iint f(x,y)\,dx\,dy = \int_c^d \left(\int_a^b f(x,y)dx\right)dy$
Double Integral in Polar Coordinates	$\iint_D f(x,y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$
Line Integral of Arc Length/ Space Curve Length	$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$
Line Integral of a Vector Field	$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)dt$ where $\mathbf{F}(\mathbf{r}(t))$ is abbrev. for $\mathbf{F}(x(t),y(t),z(t))$
Line Integral (FTC)	$\int_C \nabla f \bullet d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$
Green's Theorem(s) (Regarding the two versions *of* 'the theorem', see Figure 53)	circulation version $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA = \int_C P\,dx + Q\,dy$ AND divergence version $\iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)dA = \int_C P\,dy - Q\,dx$
Green-toward-Stokes Green-toward-Gauss (vector form)	$\iint_D \mathrm{curl}\,\mathbf{F}\bullet\mathbf{k}\,dA = \oint_C \mathbf{F}\bullet d\mathbf{r}$ $\iint_D \mathrm{div}\,\mathbf{F}(x,y)dA = \oint_C \mathbf{F}\bullet\mathbf{n}\,ds$
Surface Integral	$\iint_S f(x,y,z)dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\,dA$
Surface Integral of a Vector Field	$\iint_S \mathbf{F}\bullet d\mathbf{S} = \iint_S \mathbf{F}\bullet\mathbf{n}\,dS$
Stokes' Theorem	$\iint_S \mathrm{curl}\,\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r}$
Gauss's Theorem (aka Divergence Theorem)	$\iiint_E \mathrm{div}\,\mathbf{F}\cdot dV = \iint_S \mathbf{F}\cdot d\mathbf{S}$

Notation in **Table 6**: For the types that I call 'Green-toward-Stokes' and 'Green-toward-Gauss', I show the *curl* and *div* notation as in Stewart, pp. 1114-1115 (but with his equations flipped left-to-right to honor FTC Canonical Form, an issue discussed on page **210**).

In case you like notation with a bit more gravitas or panache, here we show an alternative style for the same two theorems, modified slightly from Salas & Hille, pp. 1085 and 1077. For a mnemonic that relates one notation to the other, see page **194**.

$$\iint_{\Omega} \left[(\nabla \times \mathbf{F}) \bullet \mathbf{k} \right] dxdy = \oint_{C} \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r}$$

$$\iint_{\Omega} (\nabla \bullet \mathbf{F})\, dxdy = \oint_{C} (\mathbf{F} \bullet \mathbf{n})\, ds$$

Table 7 may be read as a continuation of **Table 6**. As a kind of scaffolding to work from, in **Table 7** I've cobbled together my own j-dimension-in-k-dimension nomenclature with terms such as 1D-in-2D (distinct from Pure 1D), 2D-in-3D (distinct from Pure 2D), and so on. At first glance it may seem that I am attempting something distantly related to manifolds but my outlook is actually quite different (roughly the 'opposite' of how the world looks from the manifold perspective).[26] My aim here is to take the dizzying list of FTC variants and related formulas that get thrown at the Calculus III student and fit them into some kind of structure that relates *all* of them, as simply as possible, to each other. I wish to depict the objects 'from above' to provide a bird's-eye view in which we are constantly reminded: Here is a 1D object embedded *in* a 2D region, here is a 2D surface embedded *in* a 3D space, and so forth.[27] Why? Because this gives the various forms 'personality' and helps one keep track of them all.

TABLE 7: The Twelve 'Explained By' Relations, Linked By ∃

FTC Variant or Related Formula	Description/Comments ∃ = 'explained by'	Dim. Abbrev.	C: f F P: F f	The Equations, repeated from Table 6, but now flipped left-to-right in some cases
Leibniz (*the* FTC)	2D area under a curve ∃ scalars	2 ∃ 1	f F	$\int_a^b f(x)\,dx = F(b) - F(a)$
Iterated Integral	3D volume ∃ 2D area (volume of an irregular 'tower')	3 ∃ 2	NA	$\iint f(x,y)\,dx\,dy = \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$
Double Integral in Polar Coordinates	3D volume ∃ 2D disk (Figure 63 shows a 2D ∃ 2D variant)	3 ∃ 2	NA	$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$
Line Integral of Space Curve or Arc	2D area in 3D space on 1D-in-3D helix ∃ integral in t-space	2* ∃ t	f F	$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$
Line Integral of a Vector Field See Figure 63	2D area in 3D space on 1D-in-2D curve ∃ integral in t-space (for mass, work, density)	2* ∃ t	f F	$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt$ where $\mathbf{F}(\mathbf{r}(t))$ is abbrev. for $\mathbf{F}(x(t),y(t),z(t))$
Line Integral FTC	2D vector field ∃ scalars	2 ∃ t	f F	$\int_C \nabla f\cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$
Green's Theorem(s) (Regarding *curl* AND *flux*, see Figure 53.)	integral over 1D-in-2D closed curve ∃ integral over region defined by the closed curve	1* ∃ 2	(F f)	$\int_C P\,dx + Q\,dy = \iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA$ (*curl*) AND $\int_C P\,dy - Q\,dx = \iint_D\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)dA$ (*flux aka div*)
Green-toward-Stokes Green-toward-Gauss	Green's circulation version and divergence version in vector form	1* ∃ 3	F f	$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_D \operatorname{curl}\mathbf{F}\cdot\mathbf{k}\,dA$ $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_D \operatorname{div}\mathbf{F}(x,y)\,dA$
Surface Integral	integral over 2D-in-3D surface ∃ 3D	4* ∃ 3	f F	$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\,dA$
Surface Integral of a Vector Field	4D flux across 2D-in-3D area ∃ 3D	4* ∃ 3	NA	$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS$
Stokes' Theorem	4D flux across capping surface (2D-in-3D) ∃ integral over 1D-in-3D space curve OR vice versa	4*∃1* *or* 1*∃4*	f F *or* F f	$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r}$ OR $\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$
Gauss's Theorem (aka Divergence Thm.)	4D flux across 3D 'skin' ∃ integral over 3D volume	4 ∃ 3	F f	$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\cdot dV$

Legend for Table 7

Columns 1 and 5 are carried forward from **Table 6**.

Column 2 contains a characterization of each FTC variant in terms of dimensions. E.g., using *the* FTC, an object in xy-space may (surprisingly) be explained by an object in x-space. Or, borrowing the inverted 'E' to mean 'is explained by' or 'is evaluated by' or 'is solved by', we might write:

area under curve ∃ scalars

Column 3 is a restatement of Column 2 using a more succinct form, such as...

2 ∃ 1

...by way of cutting through the clutter of the Column 2 characterizations. To succinctly represent one of my hyphenated profiles, I employ asterisks. E.g., in connection with Green's Theorem, I write '1*' as the stand-in for '1D-in-2D', and so on. The letter 't' stands for t-space, i.e., parametric equations (page **221**).

Column 4 extracts the essence of Column 5 to show whether a given FTC variant has been presented there in FTC **C**anonical Form (C), with derivative f on the left and antiderivative F on the right (abbreviated as f F), or in FTC **P**ragmatic Form (P) with the antiderivative switched to the left. side (abbreviated as F f). The practice of switching sides (not my doing!) is taken up separately on page **210**.

The rows: When scanned from top to bottom, the rows of **Table 7** are meant to suggest a progression from the less complex to the more complex FTC variants (and FTC-related formulas). In this new landscape, the Calculus I/II paradigm, where a higher-dimensional integral is solved by lower-dimensional scalars, gives way to a pattern that involves integrals on both sides of the equals sign (except for the Line Integral FTC which carries the old pattern forward). And now, even when a seemingly straightforward higher-dimensional/lower-dimensional relation exists (as in Green's Theorem and Stokes' Theorem), it can no longer be taken as a guide to how the formula is exploited, in terms of H ∃ E (Hard explained by Easier). Thus, it turns out that Green's Theorem is generally used 'in reverse' because it is easier to evaluate its higher-dimensional side than its lower-dimensional side.

Regarding Stokes' Theorem, with its usage pattern of *either* C *or* P, see Stewart, p. 1141, Examples 2 and 1, respectively (see also **Verify Stokes** on page **143** below, and Schey, pp. 100-101). Possible point of confusion: From a distance, one might characterize the surface that Stokes' Theorem operates on as '2D embedded in 3D'.

True enough; however, this 2D surface is smart: It 'knows' that it resides in and bends through 3D space. Thus, as with Gauss's Theorem, the Stokes surface is defined by xyz trios of coordinates. In other words, its 2D-in-3D characterization notwithstanding, it is not analogous to the case of a balloon's surface traversed by an ant unaware of its curvature through space. The salient difference for Stokes' Theorem versus Gauss's Theorem is not dimensional; rather, it is the question of an open surface versus a closed surface.

Sidebar on Mother Nature

Note that 'in Nature', if we may presume to have an inkling of such, there is no polarity to the relationships we have been considering: In Nature, f and F are joined by a symmetrical equality, as though one were to say 'f corresponds to F' (as suggested by Figure **21** on page **42**) instead of saying 'f is explained by F' or 'F explains f'. (What would be an asymmetrical equality? 1+2+3+4+5+6 = 21 is an example. The left side contains a wealth of information showing how one arrived at the sum 21, all of which is 'lost' or 'thrown away' on the right side, thus creating an *informational* asymmetry, never mind how perfectly the numbers balance.) Only humans wish to introduce polarity into the FTC equations such that the left side is harder, the right side easier — a kind of asymmetry that would mean nothing in Nature, it seems safe to assume. Possibly this is the view of the professional mathematician as well?

Forest versus Trees

Stepping back from the 'trees' of **Table 7** to look at the 'forest':

Each FTC variant has three prominent attributes:

- a dimensional profile, usually involving n dimensions juxtaposed with $n + 1$ dimensions (in essence).
- an FTC form that is either Canonical or Pragmatic (backward)
- a hard/easier axis: something hard is solved by something easier

It is natural for the student (if not the professional mathematician) to wonder how the three attributes relate to one another. For instance, does the $n + 1$ dimension always align with 'hard' and the n dimension with 'easier'? (No. See **With the Grain and Against the Grain** below.) *Why* are some equations presented to the student in Canonical form (C) and others in Pragmatic form (P)? Does the C/P choice have something to do with the hard/easier axis? (Yes, the latter drives the former.)

Sooner or later these are questions that must arise, if not consciously then subconsciously. So I recommend that one 'pay now' by studying **Table 7** to avoid having to 'pay later' with confusion.

Partly because of inherent features and partly due to the human overlay the picture that finally emerges is messy and nonintuitive. The purpose of **Table 7** is to help one see what that picture is. If the picture seems unpleasant, Figure **63** should make up for it by showing the same elements now in their native habitat, so to speak, ordered according to their geometric identities, using the criteria described earlier.

With the Grain and Against the Grain

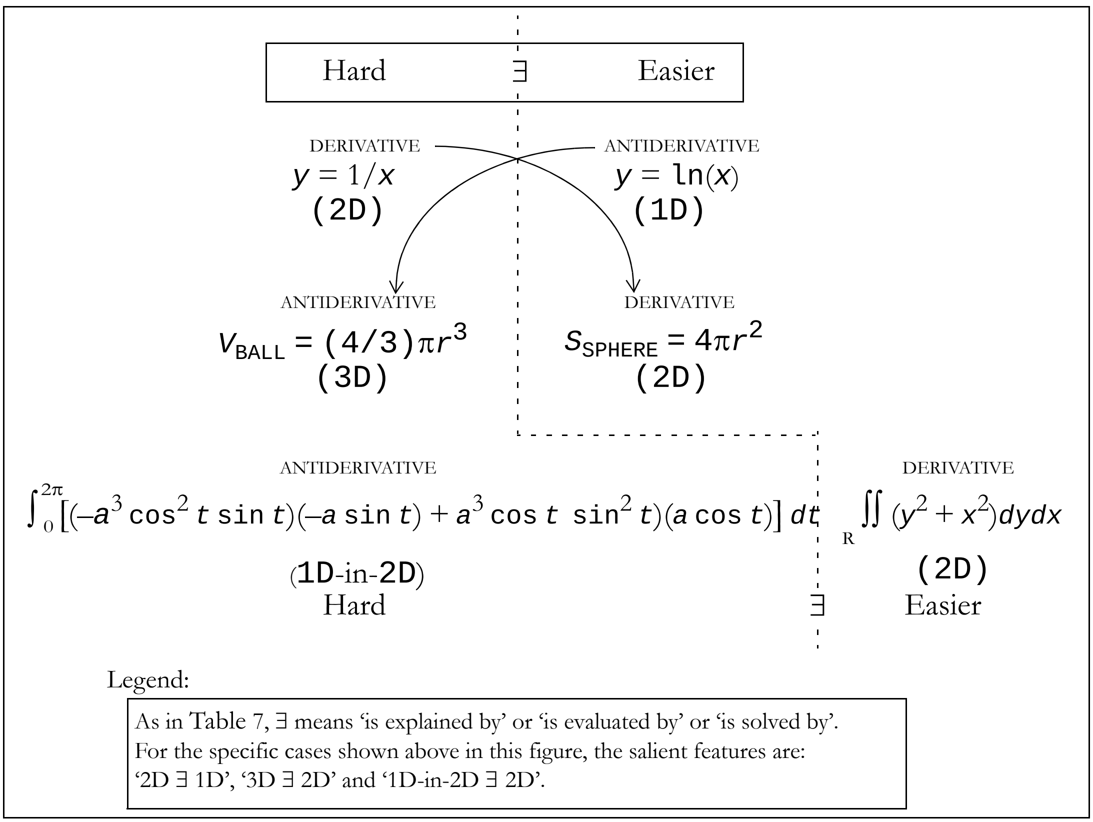


FIGURE 62: With the Grain, Against the Grain

In Figure **62**, I show several derivative/antiderivative pairs, juxtaposed with the H ∃ E paradigm (unwritten law) and with dimensional information. The first example is repeated from **Figure 22 (The Fourfold F Again, Now Aligned with 1D and 2D)**. Its dimensional profile may be said to run 'with the grain' of the H ∃ E axis, since 2D > 1D. The next example also runs with the grain of the dimensional axis, but its derivative/antiderivative pattern is 'against the grain'.

Thus, there exists a kind of dissonance between the first and second example, marked by the curved arrows that cross one another. (This same dissonance is presaged in Figure 2, where the tree plays the role of $y = \ln(x)$ and its shadow plays the role of $y = 1/x$. Thus, the derivative/antiderivative axis of the tree is at odds with that of the spherical ball, while their dimensional axes are in harmony.)

The third example (after Wood, p. 6) is presented in FTC Pragmatic Form, which forces it to go 'with the grain' as regards H ∃ E (by definition), but its dimensional profile is inherently 'against the grain' by virtue of the fact that 1D < 2D.

What does all this prove? Nothing except that the patterning of FTC attributes is not so straightforward as one might naively have imagined.

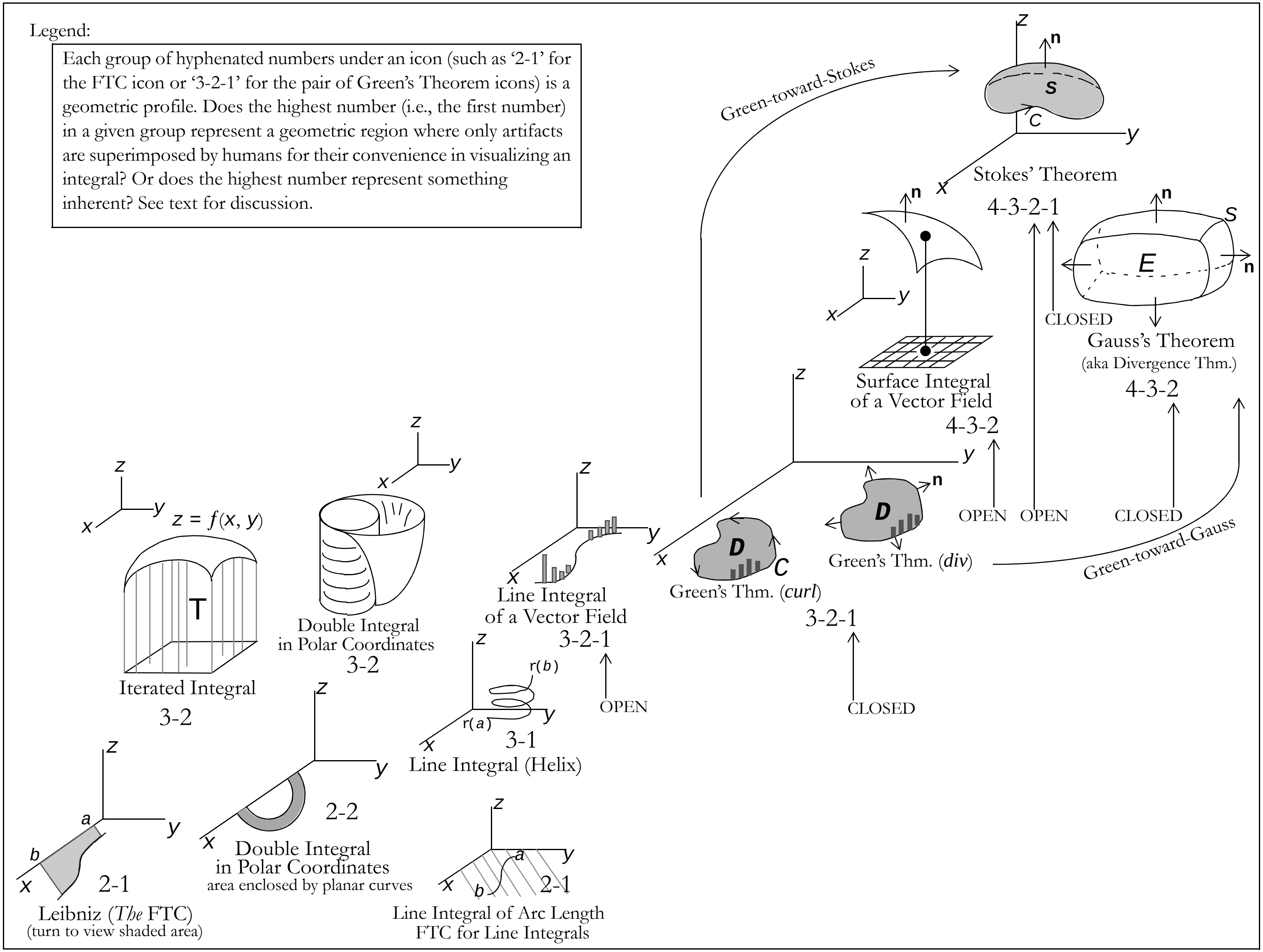


FIGURE 63: The Calculus III Landscape

Detailed Legend for Figure 63

Each group of hyphenated numbers under an icon (such as '2-1' for the FTC icon or '3-2-1' for the pair of Green's Theorem icons) is a geometric profile. Does the highest number (i.e., the first number) in a given group represent a geometric region where only *artifacts* are superimposed by humans for their convenience in visualizing an integral? Or does the highest number represent something inherent to the FTC variant (or FTC-related formula) in question? How you answer depends on your stance regarding the function versus its derivative. With the presentation in **Table 7** I acknowledged the viewpoint where a function is primary and its derivative is ancillary, in which case one is inclined to cite a lower dimension ahead of a higher dimension (like '2D-in-3D'), with the highest dimension appearing sometimes to be merely a storage place for human artifacts, so to speak. By contrast, in Figure **63**, I encourage one to try focusing on the left side because that is where the 'problem' resides (waiting to be solved by something 'easier' on the right side). By this slight shift in perspective, the highest dimension now looks perfectly real (no longer a place for 'artifacts') and one is even inclined to cite the highest dimension first. Hence the descending order in the tags, like '3-2-1'. (In the section on Green's Theorem, the discussion of Figures **59**-**61** is related to this n+1 dimensions angle on the derivative relative to its function.)

Fourth dimension: For consistency, I extend my notation scheme up to the fourth dimension, realizing that visualization of the integral becomes questionable at that point. Up there, one could try visualizing lines orthogonal to the xyz-axes, going off at a slant in some fourth direction, w. (The curl may be described as a measurement of "the circulation per unit area orthogonal to $\mathbf{F}(x, y, z)$ at (x, y, z)"; St. Andre, p. 217.)

The tags OPEN and CLOSED are reminders about the state of a curved line or curved space. By keeping track of these, we can see that Green's Theorem is an extension of the line integral, from an open curve to a closed curve (not 'an extension of the FTC' as some would say; rather, I would view the Iterated Integral as the higher dimensional analogue of the FTC). Meanwhile, the Gauss's Theorem is an extension of the Surface Integral of a vector field from an open curved space to a closed curved space. Finally, in this perspective, we see why some regard Stokes' Theorem as the most complex and central of all the FTC variants: It combines the 4-3-2 OPEN profile of the Surface Integral with the 3-2-1 CLOSED profile of Green's

Theorem.

A note about the overall physical arrangement of the icons in Figure **63**. In a general way, the progression is from lower dimensions in the SW corner to higher dimensions in the NE corner. But along the way, there are many places where I needed to finesse the position of an individual icon, either for lack of space in the implied 'grid' or to represent important information that does not line up neatly in terms of dimensional data alone, e.g., the relative 'importance' of a certain icon. (For example, the icon associated with Gauss's Theorem is the only one to *start* with 3D and 'go on from there'. By that logic, it belongs in the extreme NE corner, where it would stand out as an extreme case. But for reasons already mentioned, Stokes' Theorem may be regarded as the most complex and central of the lot, and by that logic I have placed its icon in a position that is slightly more prominent than the icon representing Stokes' Theorem.)

There seems to be a tradition of characterizing Green's Theorem as 'an extension of the FTC to the plane'. (See Protter and Morrey, p. 445. In a similar vein, Stewart says, 'Green's Theorem should be regarded as the counterpart of the Fundamental Theorem of Calculus for double integrals'; Stewart, p. 1103.) I think such assertions are misleading at worst, meaningless at best. If one were trying to build up a hierarchy of FTC variants, one would want to look first at the line integral — an integral taken over a 1D line embedded in (not 'extended to') a 2D space. Next, Green's Theorem could be called an extension of the line integral to the case with a closed curve in lieu of an open curve. That would make more sense. Next, one could look at 'area under the curve' for the FTC as the analogue to the notional 'curtain' or picket fence (Stewart, p. 1082) that stands on the line integral's open curve or on the closed curve of Green's Theorem: We may say the xy-plane as it pertains to the FTC (for showing area under a curve) is analogous to the xz-plane as it pertains to the line integral or Green's Theorem (or analogous to the yz-plane; either will do for the purpose of accommodating the 'picket fence' data in the latter two contexts). But again, nothing has been 'extended' here from one geometric realm to another. Rather, the xy-plane of the FTC is found *translated* by the line integral or Green's Theorem to the xz-plane. In terms of 3D geometry, the two planes are interchangeable.

Note that two special cases are excluded from Figure **63**:

Surface of Revolution: $S = \int 2\pi\, y\, ds$

Volume of Revolution: $V = \int_a^b A(x)\, dx$

In these two cases, the integral is evaluated directly, i.e., without a choice about using the Hard or Easier side of an FTC equation (Stewart, pp. 584 and 380). Therefore, I don't include them in Figure **63** or in **Table 7**.

Ruminations on Bonaventura Cavalieri

with Fresh Ammo Courtesy of Norbert Wiener, and Reprieve Granted by Eugene Wigner

An 'identity crisis' for the dimensions begins early, at the very bottom of the hierarchy. Consider the following definition of a point:

> A geometric **point** has no dimension — is void of quantity — and therefore cannot be drawn as "just a point". Thus, the concept of a geometric point is **axiomatic**.
>
> — Gullberg, p. 386

Returning to our terminology used in connection with **Table 7**, can there be a 'Pure 0D' object? I don't see how. To define a particular point, we envision it situated in a 1D, 2D or 3D context (at such-and-such location on the number line, or at certain planar coordinates or spacial coordinates). So we have 0D-*in*-1D or 0D-*in*-2D or 0D-*in*-3D, but no Pure 0D. The fact that doubt arises about the validity/existence of the entity at the very foundation of the hierarchy should give one pause.

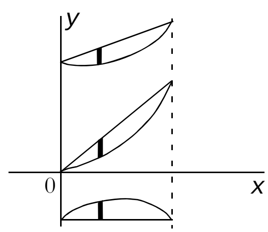
While we are down at this level of the hierarchy, it is fun to note a paradox pointed out by Norbert Wiener (in a very different context, *Cybernetics*): The probability of my aiming at and striking a particular zero-dimension bull's-eye must be p=0 (since my ill-chosen target is 'void of quantity', thus a nonentity). At the same time, my probability of hitting *some* point is incontrovertibly p=1. (I.e., my projectile must land somewhere.) Wiener himself does not use the term 'paradox' to describe this odd circumstance but in summarizing it he strongly implies one:

> Thus an event of probability one, that of my hitting *some* point, may be made up of an assemblage of instances probability zero.
>
> — Wiener, p. 46, emphasis in the original

Note in passing the wiggle word ('*may* be made up of') where we expect a direct assertion ('*is* made up of'). Thus the world of Wienerisms, characterized by passive-agressive murkiness (or 'subtlety' as he might have said in his defense). Nonetheless, the sentence in interesting. (The guy is so smart, he is sometimes interesting even when being a jerk — which is most of the time in his 'popularizing' books, and all by wily intent.) The sentence is interesting because here is someone — a highly respected mathematician at MIT — *still screwing around* with the concept that got Cavalieri (1598-1647) and Leibniz (1646-1716) in trouble centuries earlier, first with bishop/philosopher Berkeley (in 1734) and later with their fellow mathematicians: namely, the idea that lots of nothing adds up to something.

The Cavalieri Principle is summarized in terms of prisms and cylinders in Gullberg, p. 446-447, and in terms of 2D shapes in Priestley, pp. 250-251.

Very briefly, if multiple figures contain vertical segments of the same height, the areas of the figures are equal.



rough sketch after Priestley, p. 251

FIGURE 64: The Cavalieri Principle

(Bonaventura Cavalieri, a countryman and disciple of Galileo, is the one credited with writing the first textbook on integration methods, *Geometria indivisibilibus continuorum*, published in 1635, per Gullberg, p. 674.)

Evidently there is something quite compelling about the idea itself, never mind if rigorous new treatments of the calculus have long since jettisoned it from that particular realm. Notice how it arises yet again by an entirely different path in connection with **Table 7**, as we step through the dimensional profiles from 2D-in-3D to 1D-to-2D toward 0D-in-1D, and realize the whole scheme is based on an impossibility: Pure 0D as the (desired but unattainable) cornerstone. At this juncture, had we not best accept 'something from nothing'?

Briefly, here is another zero-based paradox (or inherent contradiction) with which we had a close encounter already: Take the limit on a rectangle as $\Delta x \to 0$ and $\Delta y \to 0$, as described on page **112**. By thus treating the point (x,y) as an anchor point to *approach*, we make it seem real and substantial. We circle the point 'respectfully' never proceeding to the extent of taking a flea-hop, as I call it, *onto* the point. If we did venture the flea-hop, we would annihilate the rectangle, just as surely as if the rectangle had been yanked down into a black hole, since a geometric point by definition is a nonentity of zero dimensions. How is it that a point can be at once a nonentity *and* the grain of sand around which the pearl of a limit is formed?

Now let's traverse the dimensions in the other direction: Doesn't 1D actually *need* 2D to be fully defined and existent? Then how can the simple notion of '1D' even be valid? And likewise for 2D-in-3D versus Pure 2D. And for that matter, 3D-in-4D, namely *us* embedded somehow in our mysterious 'time dimension' as the fourth, which may also be regarded as a kind of 'trap' or 'matrix' that prevents us ever from being Pure 3D creatures (if such is even possible, which seems very unlikely). In falling domino style, all the *pure* dimensions start to look dubious. But, the whole Euclidean scheme is built *on* pure dimensions to begin with (not upon our funny j-dimension in k-dimension notion), so what should we conclude? That it is all a house of cards? By 'all' I refer not just to the mathematicians' ivory tower propped up by lemmas, but to our own presumed reality.

What ties this back to the Cavalieri Principle is the concept (borrowed for the nonce from software engineering) of a 'boundary problem'. That's the common theme. In both cases, one struggles with the question of how and with what justification one might pass back and forth between the supposedly separate dimensional kingdoms. How to legitimately cross those boundaries.

Enter Eugene Wigner, to the rescue. With 'The Unreasonable Effectiveness of Mathematics in the Natural Sciences' he somewhat defuses the above issue, or at least buys the earthling species a stay of execution-by-absurdity. The point he makes is undeniable, and we must conclude that somehow the dimensions themselves *are* fundamentally valid (even if the earthling's way of *expressing* them might be provincial or wrongheaded or needlessly paradoxical). Thus, we have found something to save us, at least temporarily, from such dark ruminations on our ontology: the classic essay by Eugene Wigner!

Implicit y, Implicit z, Implicit w
and other modes of Interdimensional Sleight-of-hand

Here we pick up a thread that was begun with the discussion of Figure **40** on page **67**. When we 'get distance back', the two-dimensional aspect of this operation may seem fanciful, only an artifact of the calculus or an 'emergent property' that arises out of a situation whose essence is one-dimensional. This thread appears again in connection with Figures **59**-**61** (on pages **113**-**114**), where we note that a phantom 3D region becomes perfectly real once we've represented it on paper. We are justified in declaring it 'real' because now it may be taken as the starting point (the 'problem') for a brand-new operation: What is the area under the parabola on the arbitrary interval [2, 3]?

For coming to terms with this seemingly shadow-y aspect of the n+1 dimension, it is helpful if we return to one of the very earliest topics in precalculus: function notation. There is an aspect of it that is easily missed or forgotten because it is so obvious (and so wonderfully succinct): When you write '$y =$ ' in front of '$2x$' (or in front of any expression involving 'x') you have taken entities that are inherently 1-dimensional (values on the number line) and magically transformed them, with the stroke of a pen, into a 2D entity: a function whose graph occupies an xy-grid. Actually, the '$y =$ ' step is already implied, so all you've done is make it explicit. Thus, in the FTC, the expression $f(x)$ is already 2-dimensional, because it implies $y = f(x)$. (And this in turn makes '$f(x)$' *interchangeable* with 'y', which is why it makes sense to refer to the expression $f(x)$ as the height of an integral, while calling dx its width, as on page **196**.) By the same token, one may write '$z =$ ' in front of a 'flat' expression comprised only of x's and y's (e.g., in front of '$x + y$') transforming it with the stroke of a pen from something planar (2D) into something solid: a 3D entity (e.g., the crazily tilted plane on page **71** which arises as if by magic out of the blandness of $z = x + y$). *Except*, that's what $f(x,y)$ means *already*: $z = f(x,y)$. So you've only made the 3D aspect explicit.

Now that we've made explicit the rules of this game involving '$y =$ ' and '$z =$ ', it is natural to wonder what happens if we apply the principle to $f(x,y,z)$ (as occurs in the Surface Integral, for instance, mentioned earlier). In other words, if we write $w = f(x,y,z)$, does it mean anything? Yes. We are not necessarily in *the* fourth dimension but we are certainly in *a* valid fourth dimension now. For instance, w could be the value of the temperature as recorded at various x,y,z-coordinates all around the room, or across a geographic area of interest. Or if w tracked the location of some object in space at different times, then it could be interpreted as *the* fourth dimension. The tag '4-3-2' that I use for Gauss's Theorem in Figure **63** is another way of saying, "Remember that each *xyz* trio of values implies $w = f(x,y,z)$."

Since we think of ourselves as 3-dimensional creatures, let's look at some contrastive flavors of 'the 3rd dimension' as manifested in calculus:

- Where the iterated integral is concerned, the 3rd dimension is real (physical) and, perhaps surprisingly, we find a 3D volume 'explained by' a 2D area (a relationship that is symbolized by '3 ∃ 2' in **Table 7**).
- In the case of the 'picket fence' erected on the curved line of a Line Integral (as depicted in Figure **63**), the 3rd dimension enters the picture only indirectly and rather late in the logic, as a visualization aid: After the first two dimensions have been 'used up' in denoting the location of our data, the data itself must, perforce, be stored along some other axis to avoid clutter. That other axis is modeled as *a* third dimension, but it is not necessarily *the* third dimension.
- Suppose one were to take a 2D construct and embed it, by fiat, in the 3rd dimension, after which a phantom zero on the third axis were transformed by our calculations into a very real *non*-zero value, namely, the 'answer' to the whole problem. Now the 3rd dimension must feel somewhat real, not just a 'visualization aid' or device for avoiding clutter. (See discussion of Figure **65** toward the end of this section.)

Two special cases to exclude from the picture:

- Case A. Suppose we build a square on a line that measures 1 m long. The area of the square must be one. And if we build a cube on that same line, its volume must also be one, since 1 x 1 x 1 = 1 m^3.
- Case B. The minimalist expression $\int dx$ implicitly means $\int 1\, dx$ which in turn implies $\int x^0\, dx$. The integral is not really 'empty' after all.

In Case A, the distinction between area and volume seems to have collapsed.

In Case B, it may seem that something has emerged from nothing. Both cases are red herrings. They do nothing interesting, nothing 'alchemical' in the vein that we have been exploring. Rather, Case A may be taken as a reminder of how important units are, for without the units, yes, the area/volume distinction *would* collapse on '1'. Case B simply summarizes a notation convention, one that favors terseness (or elegance, or convenience, if you look at it in terms of less writing to do). Since x^0 has been hiding there all along, it may be integrated when the time comes as $x^{0+1}/1 = x^1 = x$.

More about units: From the standpoint of chemistry or physics, no matter what one may be up to with the manipulation of dimensions, there *must* be units that make sense through it all. This is where math and the sciences would appear to part ways (although ultimately they are closely intertwined, as argued persuasively by Eugene Wigner in the essay alluded to earlier). Consider the following passage, which Salas & Hille seem to have written without even blinking, as it were:

> The double integral $\iint_\Omega 1\, dxdy = \iint_\Omega dxdy$ gives the volume of a solid of constant height 1 over Ω In square units this [*volume*] is the *area* of Ω
>
> — Salas & Hille, p. 950 (italics added)

In other words, something flat measured in ***square*** units, yields a number that reveals the volume, measured of course in ***cubic*** units! (Just in case one thinks he/she is dreaming, a similar passage occurs in Salas & Hille, p. 945.) It is the same pattern remarked on in Figure **41** on page **69** (and elsewhere), now dialed up a notch.

Driving the point home, in Priestley pp. 269 and 271 we find the following integral evaluated first as 183 square feet, later as 183 cubic feet:

Area, volume or both?
$$\int_0^7 \pi \frac{25}{49} x^2\, dx$$

Caveat: A negative integral cannot be interpreted as a volume — at least not directly; see Stewart, pp. 1005-1006 and p. 1012, Figures 3 and 4.

In **Appendix A**, I devote a section to **The Sphere Epiphany**...

$$\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = V'_{\text{BALL}} = S_{\text{SPHERE}} = \text{Old Friend from Middle School } 4\pi r^2$$

...easily missed in the heat of the battle as one focuses on the mechanics of the Power Rule: "Gee, it works. I got the right answer!" The Sphere Epiphany is a case where taking the derivative steps us down from 3D to 2D-in-3D. Here I would like to explore a case where we move in the opposite direction, taking a 2D entity and embedding it, by fiat, in the third dimension, on the way to a derivation of the vector form of Green's Theorem (**Table 6**, page **120**, Green-toward-Stokes).

Once again, your textbook author or instructor is likely to perform this sleight-of-hand casually, letting the drama unfold silently as follows:

1. Assume a vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$, or, in scalar form, $P(x, y) + Q(x,y)$

2. Its curl is $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$

The steps above are excerpted from 'Vector Forms of Green's Theorem' in Stewart, p. 1114 (with slight rephrasing). Note the lone zero tucked quietly into the lower right corner of the matrix determinant. Where did *that* come from?

I was fortunate to have a teacher who jotted down his equivalent of our step 1 this way...

$$\mathbf{F} = P\,dx + Q\,dy + 0\,dz$$

...while murmuring to himself, 'to embed it in 3D'. Aha! That's what zero in the **k**-column means, at step 2. This in itself was a revelation: We're taking something two-dimensional and, seemingly on a whim, transplanting it to alien soil, a place with three dimensions. (My teacher did this explicitly at step 1; Stewart does it implicitly at step 2.)

David Bressoud offers a dramatization of the technique (in minimalist form, with the overhead of the *curl* excluded), where he shows not only how we can put zero *into* such a matrix, but also how we can pull something non-zero *out* of it.

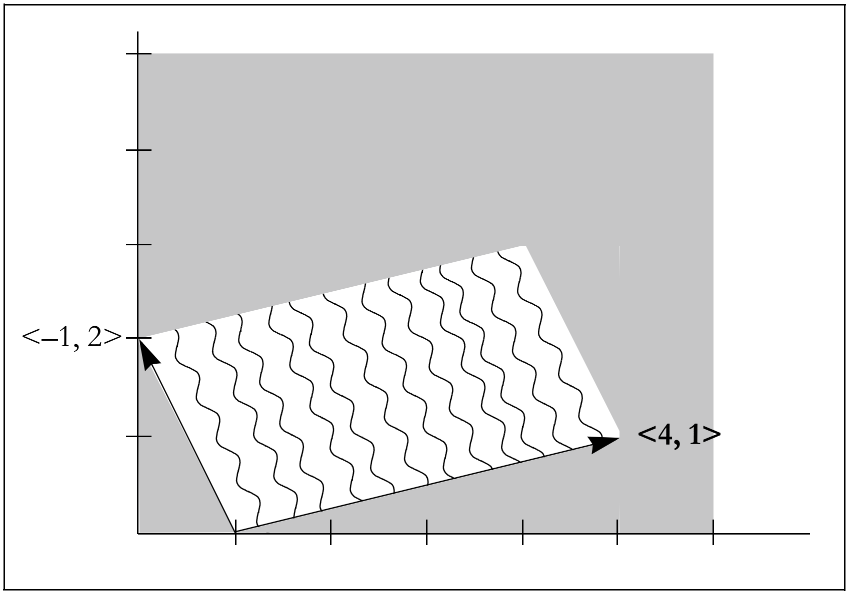


FIGURE 65: Conjuring Up a Third Dimension (after Bressoud)

In Figure 65, we imagine some sort of liquid flowing across a 2D plane. This involves a flow vector, <4, 1>, pointing roughly in the northeast direction, and a ‘window frame’ vector, let’s call it, <–1, 2>, pointing roughly toward the northwest. Our ‘window’ on the process is thus a parallelogram. Now suppose we wish to find the window’s area, to be used as a measure of, say, the flow per unit time. Noting that the corners of the parallelogram are located at (1, 0), (5, 1), (4, 3), and (0, 2), it would be easy enough in this artificial example to compute the area as 9 square units, using methods that have nothing to do with calculus. But let’s try something different. In Bressoud’s words (which echo those of my teacher, quoted earlier): “If we imbed [SIC] our vectors into three-dimensional space, then this area is precisely the magnitude of the cross-product” (p. 84). Clearly, the paired vector values <4, 1> and <–1, 2> take care of the x- and y-dimensions, but how shall we populate the phantom z-dimension to satisfy Bressoud’s whim? With a dummy value, 0, of course, supplementing each of the two vectors as follows:

$$\langle 4, 1, 0\rangle \times \langle -1, 2, 0\rangle = \begin{vmatrix} 4, 1, 0 \\ -1, 2, 0 \end{vmatrix} = (0-0,\ 0-0,\ 8-(-1)) = \boxed{9}$$

Thus seeded with zeros, the third dimension has sprouted a nine.[28] If that doesn’t send a little chill down your spine, you might want to check your pulse. (In a real-world application, one would be searching for something more elusive than the area of such a 'window' of course, whose value is evident here from grade school geometry.)

Appendix A: The 80/20 Ratio and Why There's 'Never Enough Time' for Calculus *Itself*

Preface with Mini-TOC

On page **5**, I introduced the idea of an 80/20 ratio in the content of the calculus curriculum ('80 percent algebra, 20 percent calculus'); in this appendix, we'll take a closer look at how this idea plays out for the student.

Here, to establish some perspective, I will preview three of the four examples to be presented: In **Verify Stokes**, one's natural inclination would be to focus on the new world of higher-dimensional FTC variants (the aspect that we explore at length in **Chapter VII**). But the student will soon find herself falling into a rat maze of parametric equations, matrix algebra, and trig identities, so that one grows myopic and feels the '20 percent calculus' melting away into the distance, only a brief mirage. In **Hemisphere and –x/z**, we see difficulties for the student that fall more under the heading of 'the nature of the beast': it exemplifies a certain kind of 'algebraic distraction' from the topic at hand which is so well marbled into the cake that there is no separating vanilla from chocolate after the fact. In **The Sphere Epiphany**, the ideal would be to let one's mind drift back to middle school geometry, now viewed in a brand-new light. But the math is simple, so why linger *there*? Consequently, the moment is likely lost in a mad dash forward to...(whatever is a more practical use of class time, say a quick review of umpteen trig identities).

So, there is never enough time: At one extreme, the calculus problem at hand is 'too complicated' to justify lingering on its (pure) calculus component. At the other extreme, the problem involves calculations that are 'too simple' to justify lingering on its calculus component; better do some trig identities instead. Either way, the student gets short shrift, as the pure calculus component is shunted aside.

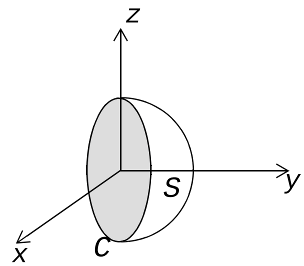
Verify Stokes

Source: The following problem and its solution are based on (but greatly expanded from) Stewart p. 1143 #15 and Clegg & Frank, p. 305 #15.

Verify Stokes' Theorem:

$$\iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$

(presented here in FTC Canonical Form)



Given:
A vector field $\mathbf{F}(x,y,z) = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$,
with a surface S whose boundary is curve C.
Specifically, assume that surface S is the hemisphere $x^2 + y^2 + z^2 = 1,\ y \geq 0$, oriented in the direction of the positive y-axis (i.e., one half of a unit ball).

Parameterization of Curve C:

$$\mathbf{r}(t) = \cos t\ \mathbf{i} + 0\mathbf{j} - \sin t\ \mathbf{k} \quad 0 \leq t \leq 2\pi$$
$$\mathbf{r}'(t) = -\sin t\ \mathbf{i} + 0\mathbf{j} - \cos t\ \mathbf{k}$$

← Took derivative using the Trig Rules on page 81.

Comment about 'Parameterization of Curve C' above: The curve C enclosing the shaded area is a circle, the 'boundary of S'. Recall, however, that the usual equation for a circle (which would be $x^2 + z^2 = 1$ in this tilted context) is *not* a function. In order to obtain a function to work with, we need to rethink the circle in terms of a *parametric equation*. Parametric equations are a major topic in precalculus. Note however that 'parametric form' as such is often just implied, not spelled out. For example, one might be given this...

$$r(t) = \langle 1 + t,\ 2 + 5t,\ -1 + 6t \rangle$$

...with the understanding that it implies all of this:

$$x = x_0 + at = 1 + t$$
$$y = y_0 + bt = 2 + 5t$$
$$z = z_0 + ct = -1 + 6t$$

For an example involving actual parametric form, see note 18 on page 235 (which is an annotation to Figure 32 on page 55).

Comment about 'right-hand side'/'left-hand side' below: First we'll evaluate the right-hand side of Stokes' theorem, which is easier, then the left-hand side. In a typical *use* of the theorem one would evaluate one side only (according to one's convenience), trusting that either side would provide the same answer, but in this case we have been asked to *verify* that handy relation, hence the need to show our work for both sides then compare them to confirm that these very different looking paths do indeed take us to the very same place.

Right-hand Side: $\int_C \mathbf{F} \cdot d\mathbf{r}$

Here we step back for a moment from Stokes' Theorem per se, and reenter an earlier phase of study where it was established for Line Integrals that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

where $\mathbf{F}(\mathbf{r}(t))$ is abbreviation for $\mathbf{F}(x(t), y(t), z(t))$

Marrying up values for y, z and x with the given function:

$$\mathbf{F}(x,y,z) = y\mathbf{i} + z\mathbf{j} + x\mathbf{k} \quad = 0\mathbf{i} + (-\sin t)\mathbf{j} + \cos t\,\mathbf{k}$$

Development of the integrand as a dot product (see page 209):

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \quad = \langle 0, -\sin t, \cos t \rangle \cdot \langle -\sin t, 0, -\cos t \rangle$$
$$= 0 + 0 - \cos^2 t = -\cos^2 t$$

Evaluation of the integral (not by a rule but with help from a published table, because it is a bear):

$$\int_0^{2\pi} -\cos^2 t\, dt = -\left[\tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right]_0^{2\pi}$$
$$= \left[-\tfrac{2\pi}{2} - 0\right] - \left[-0 - 0\right] = \boxed{-\pi}$$

The first 'half' of the problem ends here. The question is, can we obtain the same value, '–π', in the second half below.

Left-hand Side: $\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$

$\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$ means $\iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS$ in practical terms,

so eventually we'll need to define $\mathbf{n}$, but first lets tackle curl $\mathbf{F}$:

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = \mathbf{i}(0-1) - \mathbf{j}(1-0) + \mathbf{k}(0-1)$$

$$= -\mathbf{i} - \mathbf{j} - \mathbf{k} = \langle -1, -1, -1 \rangle$$

Parameterization of Surface S (in preparation for defining $\mathbf{n}$):

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho\cos\phi$$

Simplified at the right by noting that $\rho = \sqrt{1} = 1$, so ρ can be disregarded.

$$x = \sin\phi\cos\theta \qquad y = \sin\phi\sin\theta \qquad z = \cos\phi$$

$$\mathbf{r}(\phi,\theta) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle$$
$$\mathbf{r}_\phi = \langle \cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi \rangle$$
$$\mathbf{r}_\theta = \langle \sin\phi(-\sin\theta), \sin\phi\cos\theta, 0 \rangle$$

$$\mathbf{n} = \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix} =$$

$$\mathbf{i}(0+\sin^2\phi\cos\theta) - \mathbf{j}(0-\sin^2\phi\sin\theta) + \mathbf{k}(\cos\phi\sin\phi\cos^2\theta + \cos\phi\sin\phi\sin^2\theta)$$

(simplified by factoring out the Trig Identity $\cos^2\theta + \sin^2\theta = 1$, page 165)

$$= \sin^2\phi\cos\theta\,\mathbf{i} + \sin^2\phi\sin\theta\,\mathbf{j} + \cos\phi\sin\phi\,\mathbf{k}$$

Development of the integrand as a dot product:

$$\text{curl}\,\mathbf{F}\cdot\mathbf{n} = \langle -1,-1,-1 \rangle \bullet \langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \cos\phi\sin\phi \rangle$$
$$= -(\sin^2\phi\cos\theta) - (\sin^2\phi\sin\theta) - (\cos\phi\sin\phi)$$

Evaluation of the integral (next page):

$$\iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{n} = \int_0^\pi\int_0^\pi (-\sin^2\phi\cos\theta - \sin^2\phi\sin\theta - \cos\phi\sin\phi)\,d\theta\,d\phi$$

Left-hand Side (cont'd)

Evaluation of the integral:

$$\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n} = \int_0^\pi \int_0^\pi (-\sin^2\phi \cos\theta - \sin^2\phi \sin\theta - \cos\phi \sin\phi)\, d\theta\, d\phi$$

Inner integral:

$$\int_0^\pi (-\sin^2\phi \cos\theta - \sin^2\phi \sin\theta - \cos\phi \sin\phi)\, d\theta$$

$$= -\sin^2\phi \sin\theta - \sin^2\phi(-\cos\theta) - \cos\phi \sin\phi\, \theta \Big|_0^\pi$$

$$= -\sin^2\phi(\sin\theta - \cos\theta) - \theta \cos\phi \sin\phi \Big|_0^\pi$$

$$= \left[-\sin^2\phi(0-(-1)) - \pi \cos\phi \sin\phi\right] - \left[-\sin^2\phi(0-1) - 0\right]$$

$$= -\sin^2\phi * 1 - \pi\cos\phi\sin\phi - \sin^2\phi * 1$$

$$= -2\sin^2\phi - \pi\cos\phi\sin\phi$$

Outer integral:

$$\int_0^\pi (-2\sin^2\phi - \pi\cos\phi\sin\phi)\, d\phi$$

(by a Trig Identity, page 165)

$$= \int_0^\pi \left(-2\sin^2\phi - \pi\frac{\sin 2\phi}{2}\right) d\phi$$

(integrated with the help of a published table here and with chaining here:)

$$= -2\left(\tfrac{1}{2}\phi - \tfrac{1}{4}\sin 2\phi\right) - \tfrac{\pi}{2}\left(\frac{-\cos 2\phi}{2}\right)\Big|_0^\pi$$

$$= \left[-\pi - 0 + \tfrac{\pi}{4} * 1\right] - \left[0 - 0 + \tfrac{\pi}{4} * 1\right] = \boxed{-\pi}$$

Welcome to the 'sands of trigonometry' as advertised in this book's subtitle. At this point, the student may be astonished that the same value, '–π', comes out again, especially if it happens on the clock, on an exam. And rightly so, given the amount of work with paper and pencil. But what is even more astonishing is that the curve depicted in our earlier drawing (a circle in this simplest possible example) somehow 'knows' the area of the surface that bulges off into space behind it. (The formal name for the bulging part is *capping surface*, which makes sense if you picture our hemisphere in its more familiar upright position.) The short answer to this mystery is that the (generalized) integrand on the left side has been defined, with the help of untold delicate theorems, to be the derivative of the integrand on the right side (or, alternatively, the right-hand function has been defined as the antiderivative of the left-hand function, mirroring the Leibniz FTC); so 'of course it works'. But that technical explanation still leaves room to be amazed at the qualitative aspect of the relation. Or does it? This is what I'm afraid gets lost in the heat of the battle, as the student wrestles all those different flavors of computation, with only one thing really on her mind: *Will* '–π' reappear in Part 2 of the computations <gulp!> or not?

(The example above may seem extreme, although it does come directly from my own Calculus III midterm. For an example on a more modest scale whose proportions also illustrate the 80/20 concept, see Wood, page 7, Example 7. Many such examples can be found in the literature.)

Hemisphere and –x/z

Somewhere in one's introduction to the surface integral, one is likely to encounter a problem that involves a hemisphere. One is on the brink of understanding a very interesting relationship. This relationship promises to be a higher-order version of the one used to demonstrate the FTC. In the latter, something linear (F) is used to evaluate something planar (an area under the open curve of derivative function *f*). In the new relationship, for surface integrals, something planar will be used to evaluate a closed curved space 'above' it, as sketched in Figure **63** on page **127**. (How remarkable that something flat and 2-dimensional can be employed to 'comment on' something that exists as a 2D-in-3D object, as I call it. For more along these lines, see **Chapter VII**.) But one's reverie is rudely interrupted by the following two expressions (among others) that the instructor has jotted down blithely at the white board:

$$z = \sqrt{1 - x^2 - y^2}$$

$$-\frac{x}{z}$$

A moment later, one recognizes the first expression as the equation for a spherical ball ($x^2 + y^2 + z^2 = 1$), solved for z. Fair enough. This bit of transmutation is indeed part of one's current milieu: "It's in the book," more or less. But what could the second expression be? From the teacher's attitude, it seems to be something we should know. Perhaps in a secondhand book store one has picked up a copy of Schey and has noticed on page 17 that one of the partial derivatives of $z = \sqrt{1 - x^2 - y^2}$ is just that: $-x/z$ (the other being $-y/z$). Perhaps one even remembers having confirmed that computation as follows...

$$\frac{\partial z}{\partial x} z = \frac{\partial z}{\partial x}\sqrt{1 - x^2 - y^2} = \frac{1}{2}(1 - x^2 - y^2)^{-\frac{1}{2}}(1 - x^2 - y^2)'$$

$$= \left[\frac{1}{2}(1 - x^2 - y^2)^{-\frac{1}{2}}\right](-2x) = \frac{-2x}{2\sqrt{1 - x^2 - y^2}} = \frac{-x}{\sqrt{1 - x^2 - y^2}} = \boxed{-\frac{x}{z}}$$

sub back

... or maybe one was lucky enough to find same scrawled in the margin of Schey by its previous owner. (Note in passing how the computation ends on a curiously recursive-looking note, bringing z back at the last moment, strictly for reasons of aesthetics and/or convenience. This adds to its 'charm' let's say.)

Whew! Crisis averted.

Or, finding yourself with no bridge at all from $\sqrt{1 - x^2 - y^2}$ to $-x/z$, there might ensue a moment of panic as you wonder how many *other* such surprises lie in store for you on the Calculus III horizon. And so on. (A passel of students have lined up with questions at the end of this class. "...is *obe*-vee-iss," comes the refrain in a Roumanian-accented baritone. The instructor chiding others over their own $-x/z$ puzzlement perhaps?)

As suggested in the preface to this appendix, this kind of 'algebraic intrusion' into the subject is no one's fault, it's just the nature of the beast. But it can be quite aggravating at times. Well, strictly speaking, the chain of computations above is *not* algebra. Rather, it is a tightly woven hybrid of algebra *and* calculus. But subjectively, it certainly has the look and feel of algebra, and thus makes its contribution to the '80/20' notion, to which this entire appendix is just an explanatory footnote.

The Sphere Epiphany

There comes a moment in calculus that I call the Sphere Epiphany. (or 'missed epiphany' issue). It might arise in connection with a melting snowball problem, for example. (See **Figure 46 (Classic Chain Rule: Melting Snowball Derivative)** on page **85**.) It involves two formulas that one will have encountered long before the calculus class itself: the formula for the volume of a spherical ball and the formula for the surface of a sphere.

$$V_{ball} = (4/3)\pi r^3$$
$$S_{sphere} = 4\pi r^2$$

Above we see two formulas from grade school geometry. But revisiting them years later, now from the viewpoint of calculus, we discover that one is the ***derivative*** of the other (as suggested by Figure **2** on page **2**). To see this purely ***numeric*** calculus relation superimposed seamlessly on the old ***geometric*** relation is astonishing. Or it should be. But there are so many niggling details for the student to process. Assuming something like the melting snowball problem as context, one will be absorbed with questions such as the following:

'Should I be using the Classic Chain Rule or the Twisted Chain Rule?' (page **87**)

'At the outset, has the author given us a constant rate of change or instantaneous rate of change?' And so on.

Thus, the '$4\pi r^2$ moment' is likely to fly right by. Nor is the instructor likely to dwell on it, since it involves the most trivial of computations: the workhorse Power Rule (page **75**) for getting from V to V′. Rather, in that context, '$4\pi r^2$' feels like an anonymous cog in the machinery, not likely to stand out as being (also) the formula for the surface of a sphere which has 'magically' reappeared from middle school math in this utterly exotic new role. Consequently, one of the major epiphanies of elementary calculus is missed[29] (or it flits by subliminally, a mere bat in one's peripheral vision at dusk, to be revisited at a later date, ***maybe***). Rather than dwell on the V_{ball} to V′ relation, the discussion will likely turn on a dime to something 'more important':

"Oh, and by the way, here's another trig identity that you'll *really* need to know if you plan to go on to the next level of calculus!" E.g., the stuff in Tables **4** and **5** on page **81** above.

Unlike **Verify Stokes** or **Hemisphere and –x/z** above, the 'sphere epiphany' example does not connect directly with our 80/20 theme. However, it does illustrate other dynamics by which a core calculus concept may get lost in the shuffle: labyrinthine rules to learn or the gravitational pull of 'more trig identities to learn' whenever there is a perceived lull on the calculus side. (In my own case, the '$4\pi r^2$ moment' went by unremarked in class. By chance, around the same time I happened to be reviewing the V_{BALL} and S_{SPHERE} formulas in Simmons, and that was where the epiphany occurred, ironically in connection with one of my *pre*calculus books. Also, around the same time I read page 221 in Priestley, where the analogous relation of a circle's area to a circle's circumference is pointed out, and that too helped bring things into focus.)

Dead Leaf Density

A case where the earthling runs out of patience and nature runs out of granularity

There is a whole class of calculus problems that involve equilibrium states, determined with the help of a differential equation. As a representative example, I offer Dead Leaf Density, a slightly reworded and copiously annotated version of Exercise #7 in Hughes-Hallett *et al.*, p. 555. Ostensibly I will be showing what a differential equation *does* (since there is educational value in that) but my ulterior motive in this special context will be to demonstrate also what might *not* occur as one squints myopically at the labyrinthine subtleties of a differential equation: namely, an awareness of modeling discrepancies between [a] the mathematical realm, [b] the earthling realm and [c] the much larger realm of nature. In short, the very *raison d'être* of the differential equation!

Here is the essence of Exercise #7:

- Dead leaves accumulate on the ground in a forest at a rate of 3 grams per square centimeter per year.
- At the same time, these leaves decompose at a continuous rate of 75% per year.

One is asked write a differential equation for the total quantity of dead leaves at time t. After solving for the unknown function, one should sketch its graph to show how the quantity of dead leaves tends toward an equilibrium level. Also take the function's its limit to indicate more exactly the value of that equilibrium state. (If you have looked at **Appendix D** or **Appendix G**, you'll know this is not my preferred way of talking about such a situation, but I offer you here the typical textbook patter.)

We begin by writing a differential equation...

$$\frac{dL}{dt} = 3 - 0.75\,L$$

...which by definition (page **207**) contains an ***unknown*** function (L in this case) and one of its derivatives (dL/dT). The setup is deceptively simple, with only the numbers 3 and 0.75, straight from the problem statement. (Indeed, later on we'll look at a solution that *is* simple, requiring about three seconds of grade school reasoning in lieu of, say, thirty minutes of white-knuckle calculus. But first let's do it the 'fun' way, ha-ha, since many real-world situations might suggest no such Gordian Knot escape from the problem.)

This is the type of differential equation that can be solved by 'separation of variables'. In this case, that means gathering both instances of 'L' together on one side of the equation and moving the differential 'dt' to the other. From experience, one learns that this will go more smoothly if '– 0.75' is first factored out on the right side, by the following semi-tricky side-calculation:

$$3 - 0.75\,L = -\,0.75\,(-4 + L) = -\,0.75(L - 4)$$

So, the revised setup and separation of variables takes this form...

$$\frac{dL}{dt} = -\,0.75(L - 4)$$

$$\frac{1}{L-4}dL = -\,0.75\,dt$$

..after which we flag both sides of the equation for integration, the idea being that we wish to work our way from the known derivative back to the unknown function:

$$\int \frac{1}{L-4}dL = \int -\,0.75\,dt$$

Integration of the left side, by the Log Rule (page **80**), leads us to the absolute value of 'L–4':

$$\int \frac{1}{L-4}dL = \ln|L-4|$$

Integration of the right side, by the Power Rule (page **75**), turns implicit t^0 into explicit t^1:

$$\int -\,0.75\,dt = -\,0.75\,t + C$$

This is the resultant equation, putting the left and right sides back together:

$$\ln|L-4| = -\,0.75\,t + C$$

Can we pull a definition of function L out of the above equation? Not directly. First we need to get rid of '$\ln$'. Fortunately, there is a wonderful technique for doing just that. The trick is to combine exponentiation (page **210**) with the cancellation property of e and $\ln$ (page **204**), as follows:

$$e^{\ln|L-4|} = e^{-0.75t + C}$$

$$L - 4 = e^{-0.75t + C}$$

Now all we need to do is bring '+ C' into the fold. This can be accomplished by another standard operation, C-to-A conversion, as follows:

$$L - 4 = Ae^{-0.75t}$$

But what is the value of 'A'? Here we take a detour into an *initial conditions* problem. Let's roll the forest's history back to the beginning of time when there were no dead leaves at all, meaning $L = 0$ and $t = 0$, then solve for A:

$$0 - 4 = Ae^{-0.75(0)}$$

$$A = -4/e^{0} = -4$$

Now plug A's value into the earlier equation, and finally we can solve for L, the unknown function:

$$L - 4 = (-4)e^{-0.75t}$$

$$\boxed{L = 4 - 4e^{-0.75t}}$$

If we plot function L, running it out arbitrarily for 8 years, it takes shape as the curve indicated in Figure **66**. Looking at such a representation, the human inevitably feels that the curve is 'aimed at 4' and *will* touch that value at *any* moment. But this is not true, and there are other perspectives to explore.

x	y
3	3.57
4	3.80
5	3.90
6	3.95
7	3.97
8	3.99

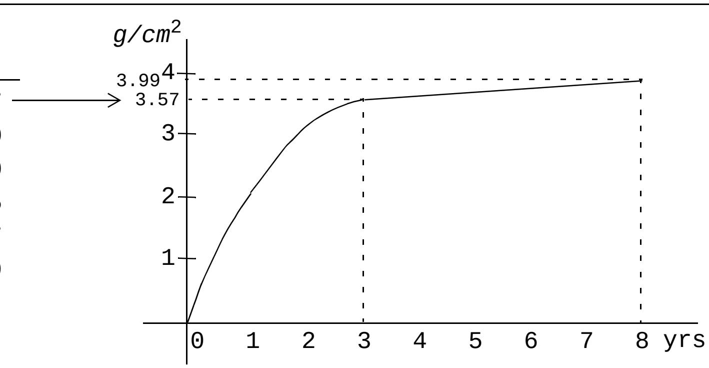


FIGURE 66: A Picture of Undecayed Dead Leaf Accumulation

As for the exercise, all that remains there is to take the limit on function L and see if it jibes with the graph.

$$\lim_{t\to\infty} 4 - 4e^{-0.75\,t} \approx 4 - 4e^{-\infty} \approx 4-(4)(0) = 4 - 0 = \boxed{4}$$

The limit is an easy one, and the answer is yes: the numbers jibe. Apparently 3.99 g/cm^2 is a reasonable representation of the system's equilibrium state. (Others will say, '4 g/cm^2 is the equilibrium state'; see comments below.)

A note about the units: Our answer, whether '3.99' or the integer 4, is not a per-year quantity, like '3 $g/cm^2/yr$' in the problem statement. Rather, it is a prediction of the forest's long-term permanent state as regards its carpet of dead leaves that are undecayed.

That is where the story of function L ends, at least in a conventional treatment. But for me that's exactly where the story *begins*. How do we actually interpret 'equilibrium state'? Stepping through values of x (for 3, 4, 5...8), when and why did we stop and make our flea-hop (as I call it) *onto* to the (supposedly sacrosanct and untouchable) limit, saying, in effect, 'enough!'? Would nature agree? Can nature keep the curve going asymptotically forever, to please the Queen (Mathematics)? If not, why not? Might nature reach an honest and *true* equilibrium point at say, 3.999987654321 after 20 years? Earthlings 'run out of patience' but nature, eventually, must 'run out of granularity' and therefore will not (cannot) continue forever to do the bidding of Queen Mathematics. In this problem, there is a gradual *increase* in density, forever; it is stated in terms of g/cm^2 which is not conducive to visualization. See the soot problem which begins on page 175, in particular Figure 77 on page 180. In that problem, the height of the soot is gradually *decreasing*. That makes it easier to visualize the problem of nature 'running out of granularity' at some point. Also, what about the Gordian Knot solution I promised, using algebra *instead* of calculus?

If we try to show exactly what is happening during the first few years of this leaf carpet's existence (as in Boyce, 2010a, where I explored the same problem to make a similar point about limits versus the Establishment), the calculations are tedious and nonintuitive. A bolder approach is to jump immediately to the end of the story and ask: What does the equilibrium state look like? After thinking about it a while, one will conclude that it must look something like Figure 67.

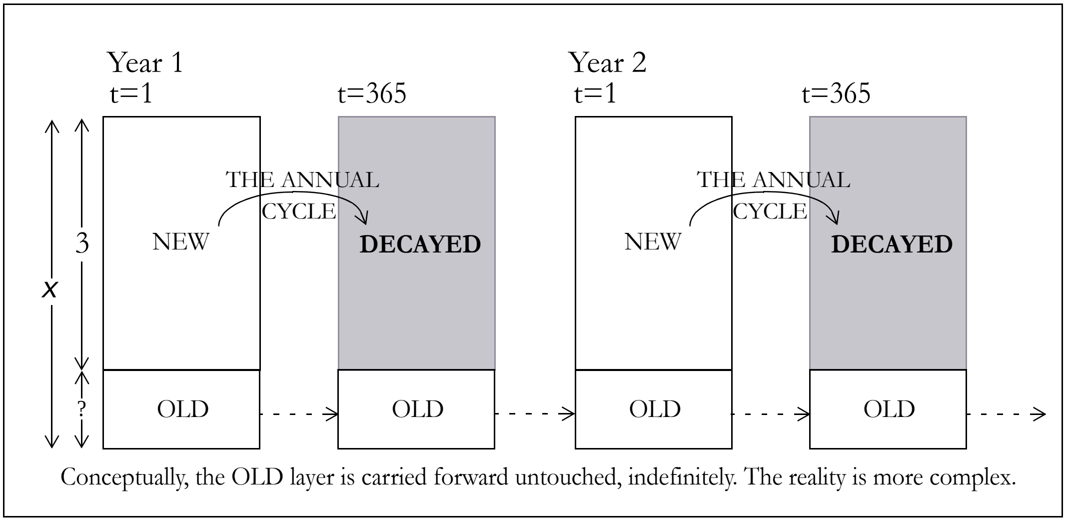


FIGURE 67: Picture of a Leaf Carpet's Equilibrium State

Prelude to Gordian Knot version

I have conflicted feelings about g/cm^2 in the exercise as given in the textbook. On the one hand, yes, in chemistry and physics, situations abound where one must be comfortable with such units, so it is good to be exposed to them in a math class too, for extra practice. On the other hand, there are subtleties at work in this exercise that are likely to be swamped by the constant 'noise' of those particular units which evoke not fallen leaves but the density of aluminum foil! In preparation for our Gordian Knot view of the problem, let's dispose of the mosquito buzz of 'per-centimeter-squared' as follows: Assume there is just *one* square centimeter (or for that matter, just one square inch or one square foot or one square meter) in the forest that we actually care about. By fiat, I say *that* is where all of this leaf-falling and leaf-decay activity takes place (for 8 years or forever). Now we proceed by talking only about *grams*!

Given:

- New leaves fall at the rate of 3 grams per year (on that special place in the forest).
- The leaves decay at a 75% annual rate.

We know that in reality, the falling and the decaying are gradual processes running in tandem. Complicated stuff. Accordingly, in constructing Figure 67, I've made a simplifying assumption about when these events occur:

- Assume that all leaf-fall occurs at once, at t = 1 each year, say January 1.
- Assume that all decay occurs at once, at t = 365 each year, say December 31.

Now we can try a rough-and-ready solution using only the logic of rudimentary algebra, as promised:

We want the new leaves to be 75% of the total, so

$$\boxed{\begin{array}{l}3 \text{ must be } 75\% \text{ of } x \\ \text{Therefore, let } x = \dfrac{3}{0.75} = \boxed{4}\end{array}}$$

Interpretation: In general, the weight of the dead but undecayed leaves found in our special place in the forest is 4 grams, all the time (except on December 31 per one of our simplifying assumptions). This appears to be an equilibrium state, and now we know all the numbers pertinent to Figure 67: 3 + 1 = 4, just as one might have guessed simply eyeballing the picture.

So, why is three pages of calculus better than the analysis immediately above, leading to the same boxed integer 4 in only a few seconds? For one thing, because the calculus reveals endless detail about the path from zero *toward* four, all along the peculiar, nonintuitive curve shown in Figure 66. In citing the Gordian Knot alternative above, I am not making fun of or casting doubt upon the calculus method. Rather, my point is that if we aren't careful, the calculus method — for all its bother and apparent rigor — winds up being just as simple-minded and naive, in its ultimate effect, as the 'Gordian Knot' version using only grade school logic.

For starts, the *equilibrium* point is not 4. That's impossible since 4 is the *limit*. Accordingly, out of respect for the asymptote, I stated the equilibrium point (at the end of the calculus version) as '3.99' not as '4'. That was not a random choice on my part. (For more about this and related issues, see **Appendix D: Imposed Limits, Inherent Limits**.) Second, there is the question of granularity (page **179**).

Summary

Strictly speaking, this dead leaf density problem may not be a case of '80% algebra and 20% calculus', as demonstrated by certain other examples in this appendix, but clearly it involves *some* kind of serious distraction from the big picture. That's why I cite it in this context. Also, because it serves double duty by introducing differential equations, the primary topic in Calculus IV, which otherwise would not have been touched upon in this book.

Appendix B: The Chain Rule(s)

There are several kinds of chain rule. The first makes its appearance in Calculus I (page **83**), in connection with composite functions. Purpose: To help differentiate a function that seems to combine two or more function types. In prime notation and Leibniz notation, the chain rule looks like this:

Prime notation	$h(x) = f(g(x))$	$h'(x) = f'(g(x)) \bullet g'(x)$
Leibniz notation		$\frac{df}{dx} = \frac{df}{du} \bullet \frac{du}{dx}$

Out of habit we are predisposed to speak of the two versions above as having 'different *notation* schemes' (prime versus Leibniz) but really we are looking at the seeds of two dramatically different implementations or expressions of the chain rule itself. So this time it's not just a matter of 'notational preference'. In Figure **68** I show how the rule works when one is thinking in terms of prime notation. In Figure **69** I show the absurd consequence of following the Leibniz notation to the bitter end. I hasten to add that the Leibniz notation is wonderful in a slightly different context: that of the chain rule with more than one variable (see **Chain Rules: More Than One Variable** on page **84**f. in **Chapter VI: Rules**). It's only in the present context that I find it rather absurd.

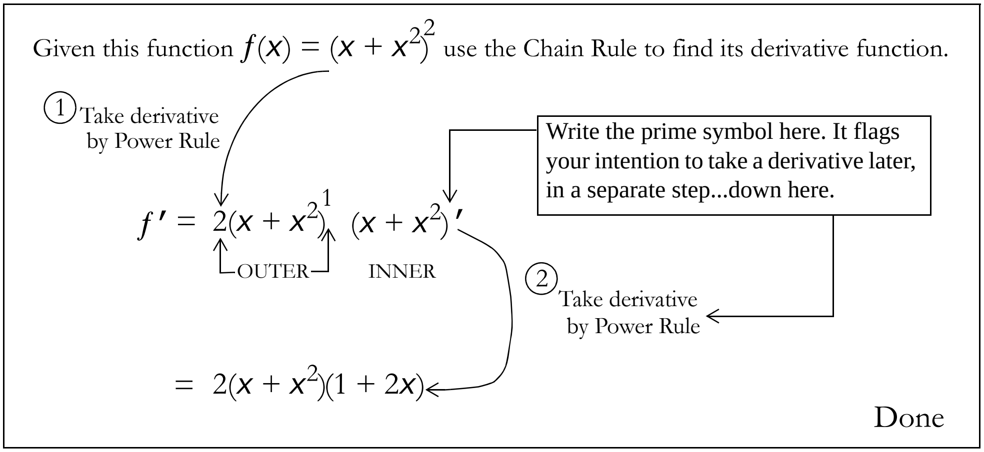


FIGURE 68: Basic Chain Rule, **Method 1:** Outer/Inner Processing

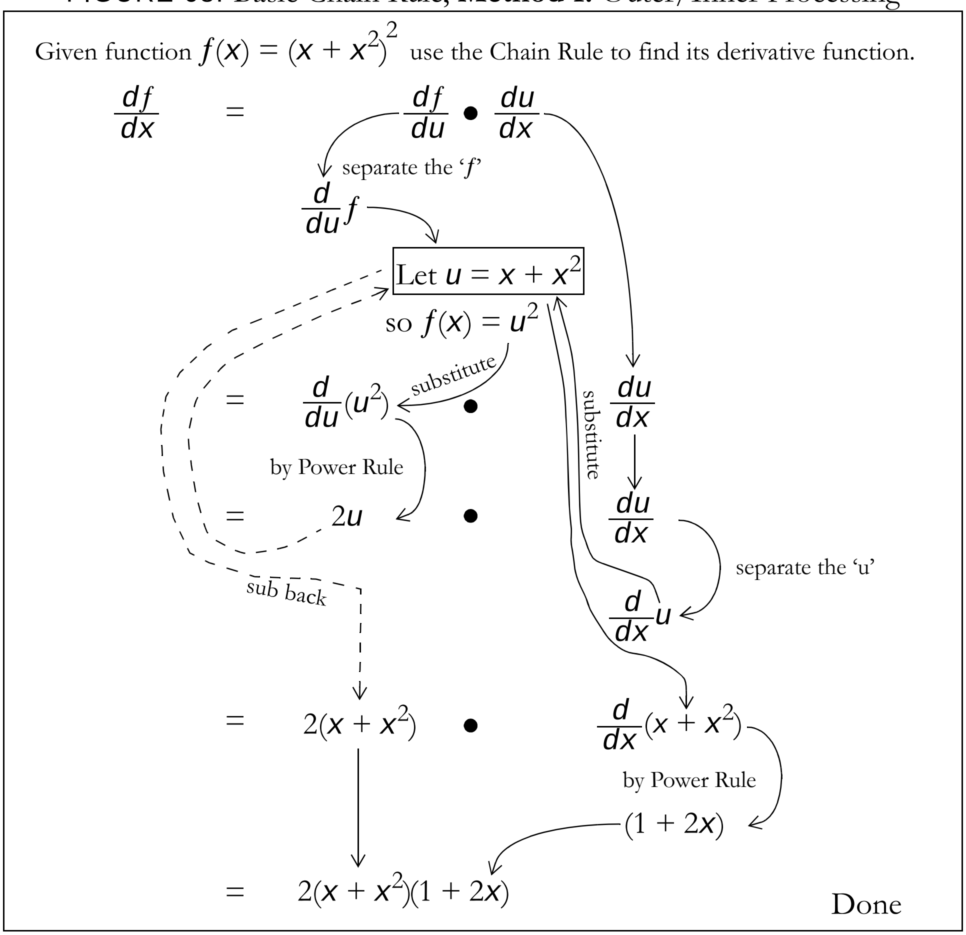


FIGURE 69: Basic Chain Rule, **Method 2:** The Chain-y Cha-Cha-Cha

Do you like Method 2? It is presented in all seriousness in such 'helpful' references as Kleppner & Ramsey's *Quick Calculus: A Self-Teaching Guide*, among many other places. In drawing Figure **69**, I have modified the format only slightly from Kleppner & Ramsey, p. 103, to bring its absurd aspect into high relief. If Method 1 was a direct flight from LA to NYC, Method 2 is the travel agent saying, "Sure thing. I have you booked for Dare Air Fight 797 via Guam and Minsk, with a layover in Dakar. We'll have you there in five days tops." Technically, the latter is still a 'correct' itinerary, and yet... It's not that the pattern `df/dx = df/du du/dx` has no value. In the next Chain Rule section, we will see it used to great effect. I see no point exercising it so frantically in the context of a single-variable situation, that's all.

An aside about Stewart's presentation of the chain rule. In Stewart's book which generally I like very much, the difference between Method 1 and Method 2 is obscured in a way that I find quite puzzling and unpleasant. By a kind of sleight of hand, he makes it appear that they are practically the same, differing only in the matter of 'notation'. (See in particular his Solution 1 and Solution 2 on pp. 178-179.) But as stressed earlier, this is an unusual case where the 'notation difference' is really a substantive difference. Seen in their true colors, Method 1 and Method 2 are radically different, as demonstrated I hope by Figures **68** and **69** above!

Hughes-Hallett *et al.* cover both methods reasonably well. The only odd thing with that committee is how they belabor Method 2 (the thorny one) for the relatively simple problems (on pp. 127-129), and reveal Method 1 (the easy one) only in connection with some difficult problems, where repeated chaining is required (bottom half of p. 129). Why? Because "It is often faster to use the chain rule without introducing new variables..." I.e., Method 1 is simply *better* than Method 2 — that's the way I would phrase it. Meanwhile, what do Salas & Hille have to say about all this? "Although Leibniz's notation is useful for routine calculations, mathematicians generally turn to the prime notation when *precision* is required" (Salas & Hille, p. 146, italics added) What? That's such a goofy non sequitur to all the above that for the nonce I must renounce Salas & Hille as my 'heros'. Put Stewart and Hughes-Hallett and Salas together (along with Kleppner & Ramsey who like to remind the reader they are two proud physicists with 'rough and ready methods', *not* fussy mathematicians), and a very peculiar picture emerges, enough to scare off any sane person from the subject. Only in calculus...

Back to Figure **68** for a moment. As a variation on its style, one may see the following:

$$\frac{d}{dx}f(x) = \underset{\uparrow\text{OUTER}}{2(x+x^2)^1}\uparrow \underset{\text{INNER (deferred)}}{\frac{d}{dx}(x+x^2)}$$

This style follows absolutely in the *spirit* of prime notation, but perhaps it doesn't quite trust the prime tick to do its job and desires something with a little more oomph? Enter d/dx, on loan as it were from Figure **69**. Either way, using the prime tick or using d/dx to flag one's intention of taking the derivative, the general method shown in Figure **68** is far cleaner and faster than the one represented by Figure **69**. By the way, the part that gets flagged by the prime symbol, or alternatively by d/dx, is what Mark Ryan calls the *glob*: the inner chunk that needs to be postponed for later differentiation — possibly multiple times if the glob is nested; see Ryan p. 75. Nice humorous touch. And in fact, believe or not, once you get past all the theory and the proofs and the attendant hocus-pocus, chaining *is* actually a fun part of calculus! Mercifully, my own teacher (page **248**) spoke only the language of Figure **68**, as it were. The horrors of Figure **69** I've experienced only vicariously after the fact, and they became one of my several motivations for writing this book: to help others steer clear of such time-wasting byways of the subject.

For more of the chain rule story, see pages **83-90**.

Appendix C: The Unit Circle Porcupine and Unit Circle Tamed

As the reader may have realized by now, the word 'on' in our subtitle ('A Calculus Oasis *on* the sands of trigonometry') was well considered. It does not promise an escape *from* trigonometry, only some new ideas that might provide temporary distractions from it. Or, in the case of this appendix, some practical trig help. (Related topic: **Trig Rules** on page **81**.)

Before setting foot in a calculus class, one must internalize the entire unit circle porcupine, as I call it:

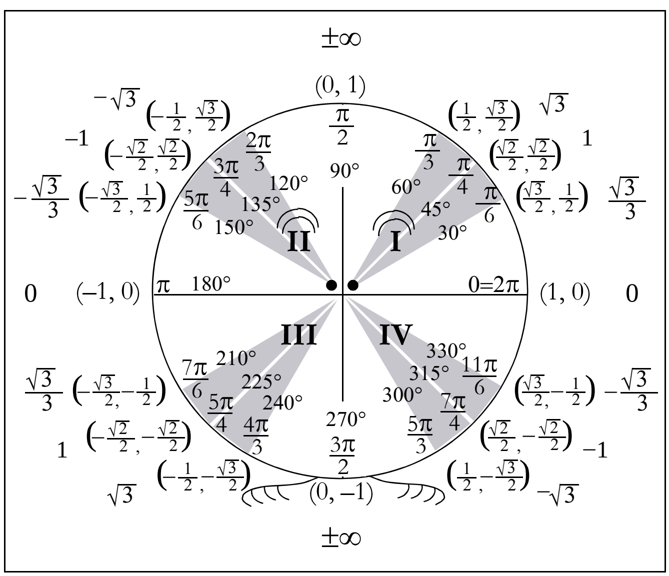


FIGURE 70: The Unit Circle Porcupine

I exaggerate not one tiny bit. Without that thing *in your head*, you won't make it through the very first quiz on Day Two. And yet, curiously, one can scarcely *find* it printed in a calculus book or even in a precalculus book. Only desultory bits and pieces of the whole tend to drift their way into print. Why? Perhaps in its full glory it is just too horrific-looking and/or deemed a waste of good ink? You tell me. (Some may find beauty in it, but to my way of thinking, it exhibits such unwavering allegiance to the sacred *pi* and *surd* symbols that the essence of the diagram is swamped in a white noise blizzard. (For more about symbols, see Appendices **F** and **G**.)

In any event, I've developed a version of the infamous 'unit circle' that some readers will find more approachable; see Figure **71** on page **164**. There I've broken it down (i.e., 'exploded' it like a parts assembly schematic) into five manageable circlets. (The graphic is to be read from lower-left to upper-right, beginning with the 'four-quadrants' circlet at the bottom.) Also, I've brought into the limelight the *cos*, *sin* and *tan* functions since they, after all, are the *raison d'être* for the whole shebang.

Usage: While I've rendered Figure **71** neatly for legibility, the idea is to use it as follows: as a guide to repeated quick *scribbling* sessions, whereby one gradually commits the whole to memory. Also, one might wish to recite certain parts of the pattern. For example, notice how all four 'trios' involving *cos* and *sin* can be recited as follows...

eight-five seven-seven five-eight

...assuming one begins each trio with the value closest to the horizontal (east-west) axis of the 'eight five' circle and disregards the decimals. Then, it's just a matter of assigning signs per the table at the bottom of Figure **71** and better values per the table that enumerates and defines the five codes. (Similarly, all tangent trios are '.58, 1, 1.7' with only the signs varying. Note the prevalence of digits 5, 7, and 8 in both mnemonic shortcuts)

Mnemonic for recreating the π radian circle in the upper right corner of Figure **71**.

Having first memorized the three denominators in quadrant I as '6-4-3', perform the three arithmetic operations shown in the following table:

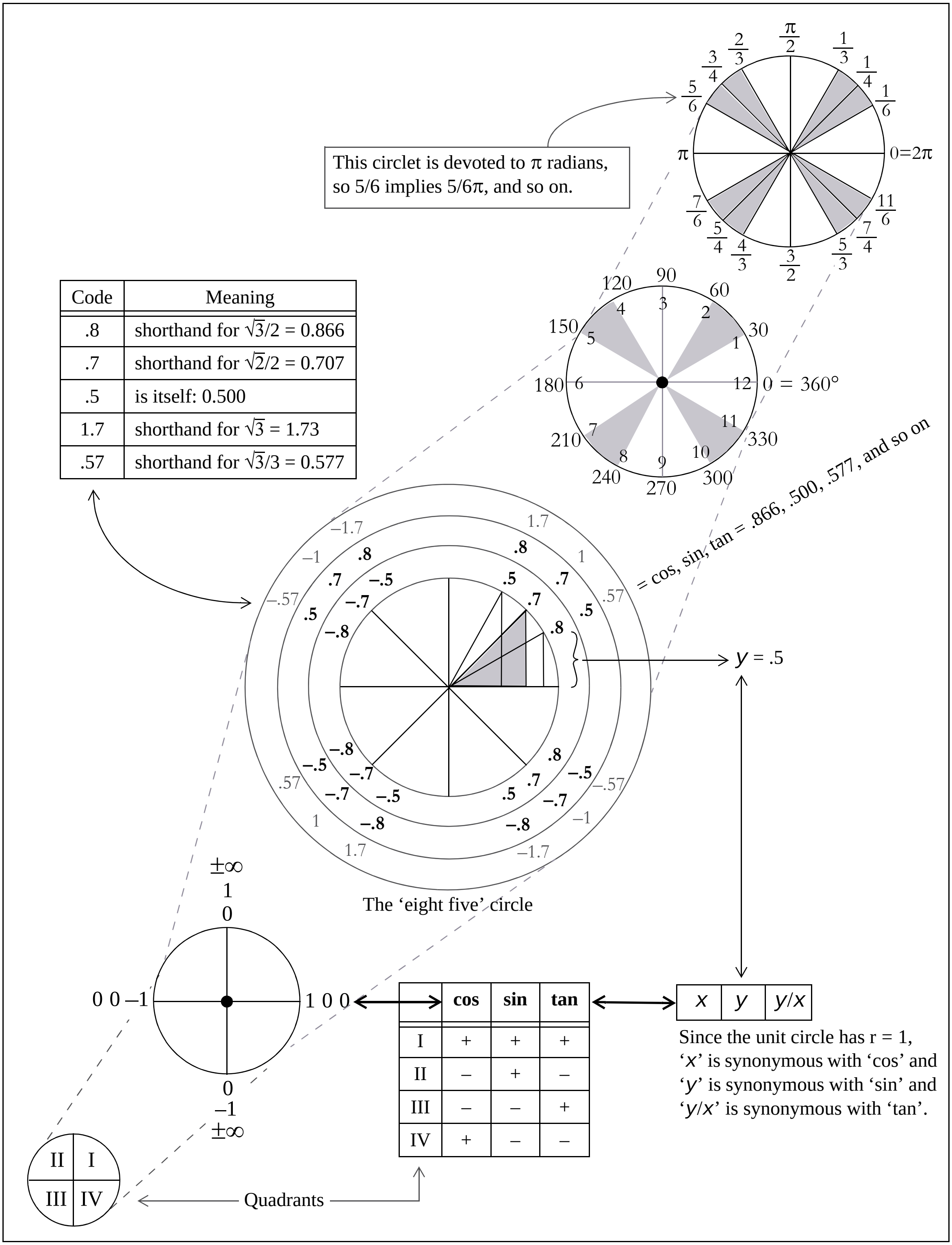
(−1)	6	4	3	
	5	3	2	
Sum:	11	7	5	(+1)
	7	5	4	

In words, decrement the initial row by one to form the second row, then sum the two top rows to form the third row. Finally, increment the top row to form the bottom row. You now have all the pieces needed to construct the fractions for quadrants II, IV, and III, respectively, of the π radian circle.

Referring to 1 o'clock and 2 o'clock (or 30° and 60°) on the 'protractor' circle, note how the triangles are drawn for cos = 8 and cos = 5 in the 'eight five' circle below. (Meanwhile, cos = 7 would be at 1:30 on the clock, or 45°.) By continuing to draw analogous triangles all the way around the unit circle for the other three quadrants, one can see why the 'eight five' pattern persists, with only the signs changing, per the table at the bottom of Figure **71**. Along the same lines, notice that the values in the upper semicircle of the 'eight five' circle (for *cos*, *sin* and *tan*) can be flipped across the east-west diameter to obtain the values for the lower semicircle, which are their mirror image, except that the signs must change for quadrants I & II versus quadrants III & IV, as summarized in the table at the bottom.

If there is an overarching philosophy or aphorism driving Figure **71**, it might be articulated this way: 'Think less, understand more'. (I.e., To the extent that one possesses a good 'crib' for quickly reproducing any piece of the unit circle as needed, one has more time remaining for observing patterns and making connections.)

Orientation difference: Note that all the `cos/sin/tan` trios in Figure **71** are to be read from an inner circle to an outer concentric circle, all the way around. By contrast, the conventional scheme shown in Figure **70** does a horizontal flip on the paired values in parentheses as you move from the eastern hemisphere (Quadrants I and IV) in to the western hemisphere (Quadrants II and III). This flip action is intended to accommodate left-to-right reading of a pair, but all it does is introduce unneeded dissonance to the picture, I think. My arrangement permits 'radial' reading instead, in a uniform manner for all three values `(cos/sin/tan)`, consistently at all points of the compass.



Code	Meaning
.8	shorthand for $\sqrt{3}/2 = 0.866$
.7	shorthand for $\sqrt{2}/2 = 0.707$
.5	is itself: 0.500
1.7	shorthand for $\sqrt{3} = 1.73$
.57	shorthand for $\sqrt{3}/3 = 0.577$

	cos	sin	tan
I	+	+	+
II	–	+	–
III	–	–	+
IV	+	–	–

FIGURE 71: The Unit Circle Tamed

Trig Identities

TABLE 8: Trig Identities

Primary Forms	Derived Forms	Name/Comment
$\cos^2 x + \sin^2 x = 1$	$\sin^2 x = 1 - \cos^2 x$ $\cos^2 x = 1 - \sin^2 x$	Pythagorean Identities
$1 + \tan^2 x = \sec^2 x$	$\tan^2 x = \sec^2 x - 1$ $\sec^2 x - \tan^2 x = 1$	
$1 + \cot^2 x = \csc^2 x$	$\cot^2 x = \csc^2 x - 1$ $\csc^2 x - \cot^2 x = 1$	
$\cos 2x = \cos^2 x - \sin^2 x$		Double-angle and 'Half-angle' Identities
$\cos 2x = 2\cos^2 x - 1$ $\cos 2x = 1 - 2\sin^2 x$	$\cos^2 x = (1 + \cos 2x)/2$ $\sin^2 x = (1 - \cos 2x)/2$	(These two can be employed as *Power Reduction* identities.)
$\cos\theta \sin\theta = \frac{1}{2}\sin 2\theta$	$\sin 2\theta = 2\cos\theta\sin\theta$	

Table 8 (Trig Identities) contains the minimal list required for 'survival'. Note that the rudimentary trigonometry functions for sine, cosine and tangent are covered indirectly by Figure **71** above.

Notation: Why do I write '$\cos^2 x$' not '$\cos^2 \theta$'? The point is recitation *and* 'scribbling' of the identities: The *theta*-form would be too awkward for use in one's private recitation or written review at a fast scribbled pace. It is enough that one must invert the superscript and variable, looking at '*cosine squared ex*' but saying '*cosine ex squared*' (because that's what it means). When the time comes to write one of these expressions properly, in homework or on an exam, you will remember that really 'x' means 'θ' and sub it back in by second nature. (Explanation repeated, in essence, from **Trig Rules** on page **81**.)

Two more, just for fun (and as a reminder that there are many more such identities to learn):

$e^{ix} = \cos x + i \sin x$ (This is closely related to Euler's formula $e^{i\pi} = -1$; see Gullberg p. 507.)

$c^2 = a^2 + b^2 - 2ab \cos \theta$ (This is the Law of Cosines, which would be my candidate for 'the most beautiful of all'.)

For students of Calculus III, there is a table in St. Andre's *Study Guide*, p. 112, that is very useful (as in 'worth the price of the book' perhaps?) His table is a kind of 'cartesian product' that forms an appealing matrix for locating the 18 equations needed for converting between rectangular, cylindrical and spherical coordinates.

Proposed enhancement (personal preference): Among those 18 equations, we see two instances of the following pair: $\tan(\theta) = (y/x)$ and $\cos(\phi) = (z/\rho)$. To my way of thinking, the table works even better if we rewrite those equations to isolate θ and ϕ on the left side of the equals sign this way: $\theta = \arcsin(y/r)$ and $\phi = \arccos(z/\rho)$. Note that the denominators are now r and ρ, not x and ρ. So there is a small price to pay for this arrangement: one must first solve for r. But this 'price' is a light one since typically one would have occasion elsewhere to solve for r anyway, as $r = \sqrt{x^2 + y^2}$.

Round-tripping 45 Degrees Through a Calculator in Mode Radians

In Figure **71** on page **164**, I promote thinking in terms of so-called 'approximate' values such as 0.707 along with their 'exact' counterparts, such as $\sqrt{2}/2$. In that context, the motivation is twofold: (1) to internalize the unit circle in a form that is faster and more practical than Figure **70**, the traditional scheme; (2) to internalize the unit circle in a way that provides more understanding than the traditional way. (For a general discussion of 'approximate' and 'exact', see page **204**.)

Beyond all of that, there is a point to make regarding calculators. Suppose you have occasion to 'round-trip' the value 45° or 90° through a calculator, with mode = = radians. In Figure **72** I have drawn a schematic of the process, using an arbitrary icon to represent several of the calculator's trigonometric functions as 'black boxes' with input on the left and output on the right:

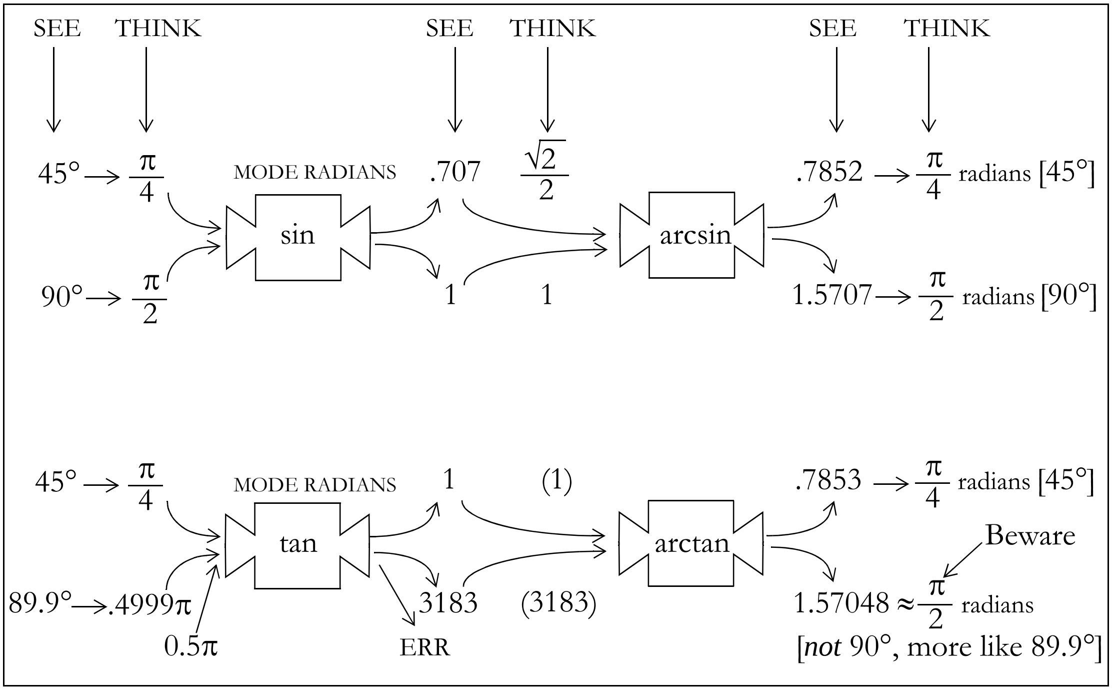


FIGURE 72: Round-tripping 45 Degrees and 90 Degrees

With the `tan` and `arctan` functions, the round-tripping challenge becomes even greater: There we introduce an arbitrary value close to one-half *pi*, to exercise the 'tail' of the function. Also, we enter one-half *pi* itself, to show that it is undefined for the `tan` function (ERR). On the `arctan` side, we show how easy it is for a misunderstanding to arise in this case if one accepts the output value as the decimal equivalent of one-half *pi* radians, ignorant of its 'history' that began with 0.4999 *pi* as input to the tan function. The point I wish to convey with Figure 72 is that one must think constantly in both languages (so-called 'exact' *and* 'approximate'), not just the former as advocated by the establishment. Also, Figure 72 serves double duty as an example of the 'white box' concept, to supplement the discussion of Figures 12 and 13 on pages 23-24.

Appendix D: Imposed Limits, Inherent Limits

The limit plays a crucial role in the very definition of calculus, as indicated in the passage that we quoted from page 1 in Salas & Hille, on page 10 above. Here they are again, underscoring the point at the beginning of their chapter on limits:

> *Without limits calculus simply does not exist. Every single notion of calculus is a limit in one sense or another.*
> What is instantaneous velocity? It is the limit of average velocities.
> What is the slope of a curve? It is the limit of slopes of secant lines.
> [and so on] — Salas & Hille, p. 47 (their italics)

Thus, in the traditional curriculum, 'limit' is presented as one thing, a thing of supreme importance, worthy of great respect. The point I wish to make in this appendix is that the reality is very different from how the traditional curriculum presents this topic. The reality is that limits play two very different roles in calculus, only one of which involves the tiptoeing respect described in Chapters I and II.

In Calculus I, we are introduced to what I would call the *imposed limit*. In Calculus II, the focus shifts (without warning or comment) to what I would call the *inherent limit*. When we look at a collection of rectangles fitted under a curve and estimate the area by taking the limit on the sum of innumerable such rectangles (as $i \to \infty$), that is what I mean by an imposed limit. I call it 'imposed' because it has nothing to do directly with the curve itself. It is an analytical device superimposed on the curve by humans to discover its area. (Similarly, the idea of instantaneous velocity involves imposing the limit as $h \to 0$ on a formula that is essentially the Difference Quotient introduced on page 27; e.g., in Stewart, p. 118.) By contrast, when we estimate the terminal velocity of a skydiver, the limit is *inherent* in the behavior of any falling object in nature. Drop a stone for eternity, and the limit on its velocity will manifest itself automatically to anyone in a position to simply observe the stone's descent for, say, one tenth of an eternity. Meanwhile, however, there is no such thing as the stone's 'terminal velocity'. The term is nonsensical (except to humans). Here is how the situation develops: The human becomes bored after a while with 'increments in velocity that are too small to matter' and at this magical moment (defined by 'when I feel like it') the human usurps the limit by stealing its value and tacitly assigning that value to the expression 'terminal velocity', thus replacing something real and significant in nature by something unreal and trivial. In some presentations, the limit's role in the shell game described above is at least acknowledged, this way:

"The *terminal velocity* of the raindrop is $\lim_{t\to\infty} v(t)$" (Stewart, p. 635, italics in the original). In other presentations, the limit is not even mentioned. For example, in the following quote the authors are clearly thinking about the limit, but they refer to it instead as the 'horizontal asymptote': "The horizontal asymptote represents the *terminal velocity*, mg/k" (Hughes-Hallett *et al*., p. 553, italics in the original). Note that their assertion is false. The horizontal asymptote is the limit; it does not *represent* anything other than itself, certainly not the chimeric 'terminal velocity' of an earthling. (The term 'terminal velocity' is insidious. It is not exactly a misnomer, but it is a term that is guaranteed to be misinterpreted as 'maximum [possible] velocity' by 99.9% of those first encountering it, and by 99% of students even one year hence. Thus, however speciously 'accurate' the term may have been in the mind of the person who coined, it is de facto a misnomer.)

My nickname for the maneuver described above is the 'flea-hop': Picture a human flea judging the diminishing gap between the function and its horizontal asymptote (aka its limit). When the gap is 'small enough', the flea hops from the function (where it belongs) onto to the limit, supposedly sacred ground as presented in Calculus I. In Calculus I, we are reminded to *keep* writing $\lim_{h\to 0}$ before every line of one's calculation on an exam paper (or lose points!) *That's* how quasi-sacred it is. Also, we are warned not to actually *assign* the value zero to h too soon when using the difference quotient to find a derivative function. *That's* how delicate the operation is. Both of these aspects are illustrated in Figure 16 on page 29. In fact, they are two sides of the same coin: Only when zero has been assigned may one stop writing the $\lim$ symbol. In short, in Calculus I we treat the limit with great respect, and with literal and figurative circumspection. How can one not be taken aback by the contrast of Calculus II where the limit is defiled routinely by silent unacknowledged flea-hops?

In Calculus II, the term 'limit' recurs, but now it has a very different flavor. Now it is something we seek when integrating certain functions on the interval 1 to ∞. In this context, the first question is: Does a limit even exist? If a limit does not exist, the function is declared *divergent*. (The term 'divergent' is a semantic trap; see page 208.) If a limit does exist, the function is declared *convergent*, because it converges on the limit. (For more about these definitions, see Stewart, p. 558.) But what exactly is the limit's value? That's the next question.

In Calculus II, this kind of limit (one that is sought rather than declared) is generally not treated with respect. To the contrary, it appears only to be a sort of gimmick, a technique (the flea-hop, page 34) for relieving the human of tedium, the bane of all macroscopic beings (trapped as we are in-between the cosmic scale and the atomic scale where the significant action is). The trouble is that Calculus II attempts to shoehorn limits into the realm of *substances*-with-quantities whereas the essence of limits involves *in*substantial time, namely eternity. There ensues the ultimate mistake in mixing 'apples and oranges', with embarrassing consequences as illustrated in Figure 73. (As for why the 'flea-hop' of Calculus II is condoned, I would assume it is a symptom of the pro-technology anti-science tilt of academia generally. The engineering student, I admit, has every right to insist on the flea-hop, for practical reasons. But that issue should not be allowed to muddy the waters of the calculus curriculum. It should be handled inside the walls of the engineering department.)

Another semantic trap: When talking about these *improper integrals* that come in two flavors, *convergent* and *divergent*, it feels natural to say the corresponding *function* is convergent or divergent (as I did above), but it is a lie. What we mean is the function has properties such that the *integrated area under its graph*, when plotted from 1 toward ∞, is convergent or divergent, not the function itself, which simply keeps plodding along, taking values in on the x-axis and putting them out on the y-axis, like any other function. (Or, the reference might be to its volume, when its area is taken through a notional rotation to make the function's output three-dimensional. Also, as a practical matter, the interval along the x-axis is typically shown as 1 to b where b is any suitable proxy for ∞, i.e., some 'really big number'.)

If we really respected limits (the way we seem to in precalculus and Calculus I), we would refrain from using the term 'finite' in connection with a convergent integration and speak instead of an integration that is Convergent For Eternity or CFE — something like that. This would serve as a constant reminder that the process itself keeps going forever, and we assign it a fixed value such as '2.468' only out of human boredom, i.e., the inherent catastrophic deficiency of a macroscopic being in trying to deal with time or space on the cosmic scale or atomic scale, where everything important transpires. (Because the velocity of a falling object is Convergent For Eternity, there can be no such thing as ‘terminal velocity’. The expression is meaningful only to an impatient earthling; to Mother Nature it would be a kind of gibberish.)

And in lieu of using the term ‘infinite’ in connection with a divergent integration, we would speak instead of an integration that is Divergent For Eternity or DFE — something like that. This would serve as a constant reminder that the process itself keeps going *for* eternity (not *to* infinity) and we assign it the fixed pseudo-value ‘∞’ only out of human boredom, i.e., the inherent deficiency of a macroscopic being in dealing with time or space on the cosmic scale.[30]

In Figure **73**, I offer an extended example to illustrate the difference between [a] what humans like to talk about versus [b] what nature actually does or ‘cares about’. The contrast is dramatic.

Nomenclature note: By now, I hope it is clear that I do not mean ‘imposed limit’ in a pejorative sense (nor ‘inherent limit’ in a condoning sense). In fact, the legitimacy/illegitimacy of the two types plays out the other way around: Because of the way limits happen to be used in Calculus I and Calculus II, it is the ‘imposed limit’ that is the legitimate type, while the appearance of an ‘inherent limit’ spells trouble.

So far I’ve used the tags ‘Calculus I’ and ‘Calculus II’ as a convenient way to highlight two different attitudes about limits. I have not mentioned Calculus III (vector calculus) yet. In Calculus III, along with the inevitable flea-hop of Calculus II, one may also encounter limit operations that are once again legitimate, with a flavor like those of the limit as first introduced in Calculus I. For an example, see pages **101**-**117**.

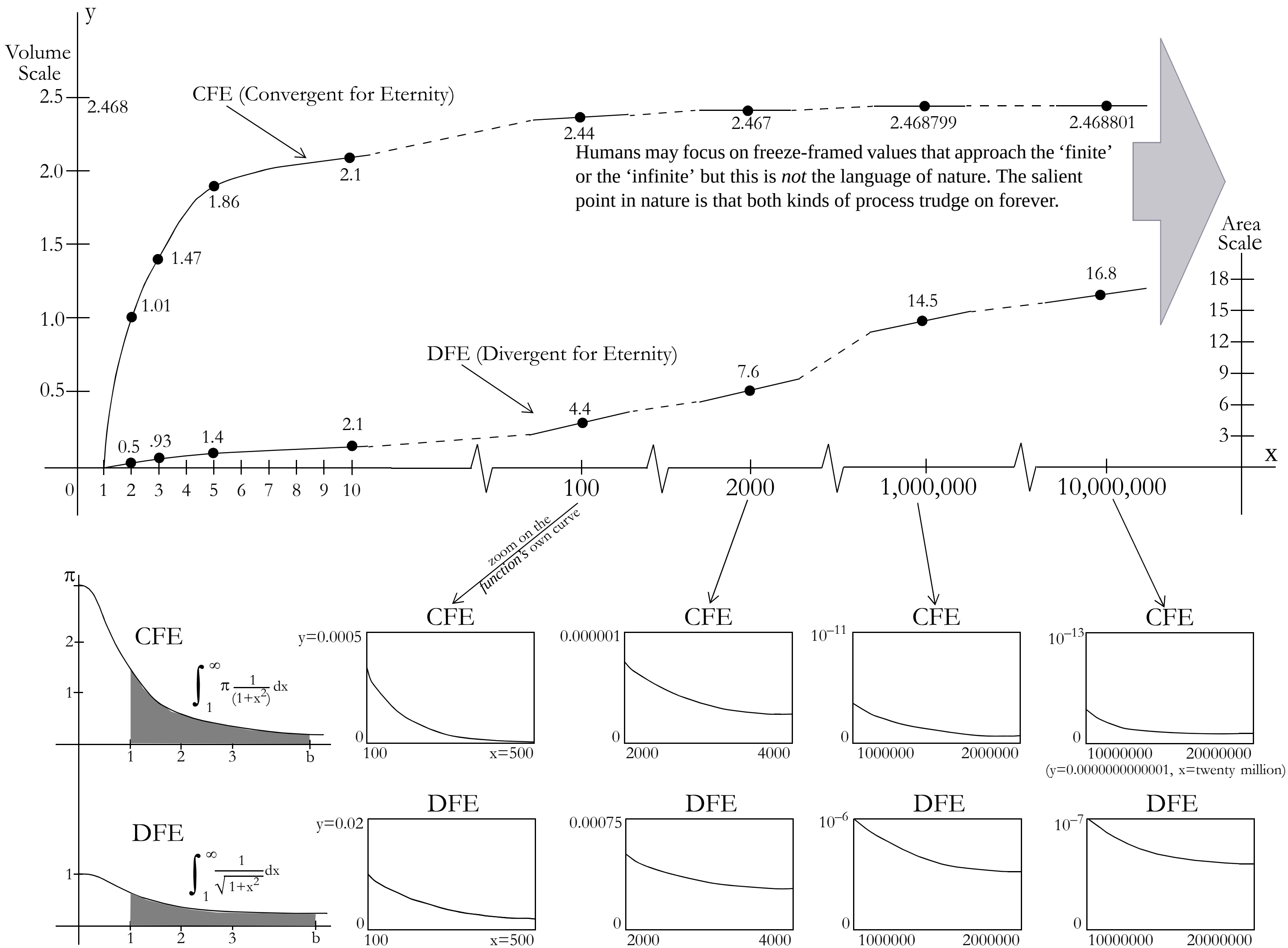


FIGURE 73: Convergent/Divergent Also Means Eternal/Eternal

In Figure 73 I use separate y-scales as follows:

[a] a Volume Scale to handle the 'finite volume' represented by the upper curve (for integrating $\pi(1/(1+x^2))$ over the interval [1,∞] on the x-axis)

[b] an Area Scale to handle the 'infinite area' represented by the lower curve (for integrating $1/\sqrt{1+x^2}$ over the interval [1,∞] on the x-axis). Conventionally, this scale would 'point to infinity' but I truncate it at 18 by way of emphasizing the x-axis (large grey arrow) over the y-axis.

My reason for using separate y-scales: This allows one to highlight the psychology of wrestling the large number on the x-axis (which extend arbitrarily to 10,000,000).

From the human perspective, the upper curve approaches its limit 'too subtly' and therefore soon becomes a bore. The solution (for humans): Arbitrarily halt the process and declare its type 'convergent' and its outcome 'finite'. When? Whenever I *feel* that I've reached the limit of my patience. Say at $x = 10000000$ and $y = 2.468801$, for example. That seems like a fine place to stop. (I.e., a good place to perform the *flea-hop*, as I call it: from the function onto the limit itself.)

From the human perspective, the lower curve is approaching some extremely large number 'too slowly' and therefore soon becomes a bore. The solution (for humans): Arbitrarily halt the process and declare its type 'divergent' and its outcome 'infinite'. When? Whenever I *feel* that I've reached the limit of my patience. Say at x=10000000 and y=16.8, for example. That seems like a fine place to stop.

Above I speak of 'the human perspective'. What other perspective am I implying? The hypothetical perspective of nature or the cosmos or a machine or a space alien or God. From any of those nonhuman perspectives, these are both processes whose essence has just as much to do with time as with quantities. Note the phrase 'soon becomes a bore' which occurs twice above. Meanwhile, on the cosmic side, both processes go on forever: their shared feature is eternity — something sharply in focus, not fading out as something 'boring'.

My dual scales for the y-axis also help illuminate the Tortoise and Hare aspect of Figure 73: The CFE function is the Hare, sprinting into the neighborhood of 2.468 something, the DFE function is the Tortoise, trudging toward infinity on the y-axis but *both keep on forever on the x-axis.* Depending how you look at it, there is something heavenly or hellish in its contemplation.

To humans, a downward sloping curve implies 'almost at the bottom'. To God or

Nature or the cosmos, a downward sloping curve means no such thing. The eight 'windows' at the bottom of Figure **73** provide this other, extrahuman perspective: curvature (growth, decay?) is *forever*. Although we use numbers in Figure **73** to illuminate the pattern, the true subject is *time*, not quantities.

One may say that the Holy Grail of calculus is the *flattened curve* — the curve approached so closely that it becomes indistinguishable from a straight line (its tangent at that chosen point). Not to dispute the usefulness of that general philosophy, but what I am trying to show in Figure **73** is roughly the opposite idea: That no matter how incredibly far out you go on the tail that seems — from one perspective — to be flattening out, the curvature is in fact alive and well,[31] continuing for eternity. If you take some time to study *all* of Figure **73** you'll see that it is a very strange animal indeed.

Modeling an Extremely Flat 'Wafer' of Urban Soot

Bringing it home to a more practical situation, we look next at an example that is literally 'on the ground': A model of urban soot over a circle of 5 km radius.

Given: Soot spreads from an urban source over a circular area with a 5 km radius. The density function for the soot is $y = H(r) = 0.000000115e^{-2r}$ where r is the radius in kilometers. Write the definite integral and evaluate it. (This problem statement is after Hughes-Hallett[32]; its elaboration in Figure **74** is my own.[33])

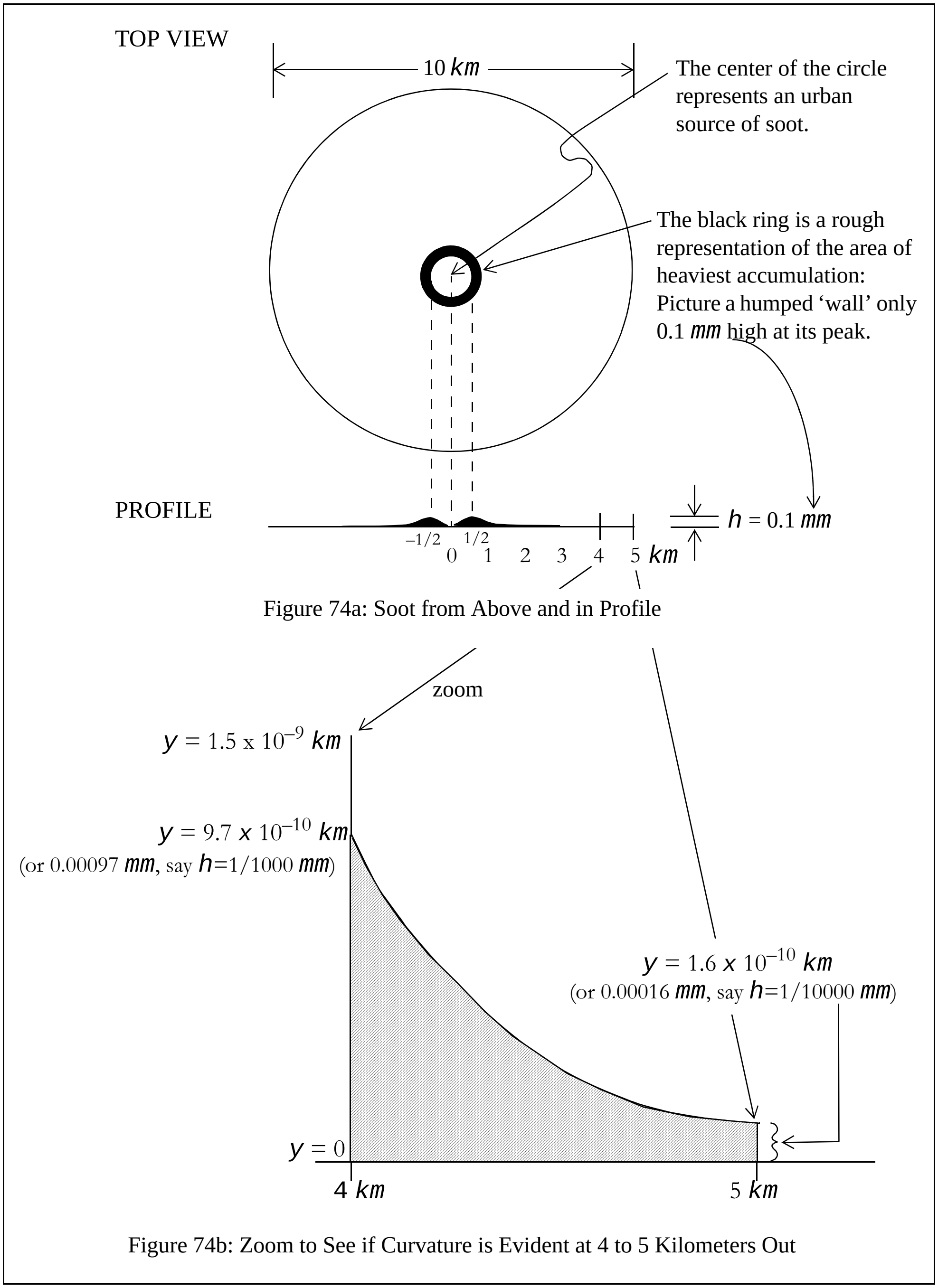


Figure 74a: Soot from Above and in Profile

Figure 74b: Zoom to See if Curvature is Evident at 4 to 5 Kilometers Out

FIGURE 74: Modeling an Extremely Flat Wide 'Wafer' of Soot

The integrand is 0.000000115 e^{-2r} km $2\pi r$, and this evaluates to 181 cubic meters of soot. (The computation is rather involved, but underlying everything is the simple concept of mass = density * volume, backed out from the definition of density as mass/volume.)

For a better understanding of the double hump seen in Figure 74a, we'll begin by looking at just one of the humps, displayed in a window[34] whose dimensions run from $x = 0$ to $x = 5$ kilometers and from $y = 0$ to $y = 10^{-6}$ *km*. See Figure 75.

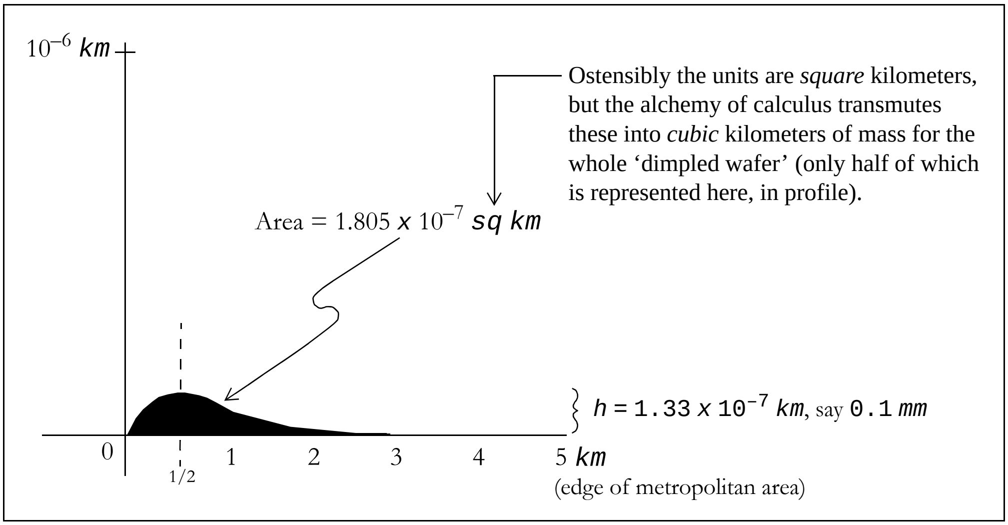


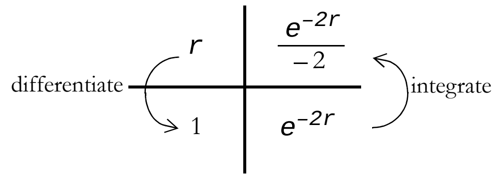
FIGURE 75: The Circular Wall of Soot in Profile, Closer View

One integrates the 2D ***area*** under a curve to obtain a value 1.805×10^{-7}. But this value turns out to be the answer to a 3D question: the ***volume*** in cubic kilometers of the accumulated soot mass! (Later, to bring it back to the human scale, we convert 1.805×10^{-7} km^3 to 181 cubic meters.) Thus, it is analogous to the case in Figure 20, where a ***linear*** number provides the answer to a ***spacial*** question — now with the two dimensions cranked up a notch.[35]

Next, some details of the soot problem solution. The technique shown on the right side of Figure 76 is called Integration By Parts (introduced on page 95).

Evaluate

$$1.15 \times 10^{-7}\,(2\pi)\int_0^5 e^{-2r}\, r\, dr \qquad \int u\,v' = u \bullet v - \int u'\,v$$



$$\int u\,v' = r\,\frac{e^{-2r}}{-2} - \int 1 * \frac{e^{-2r}}{-2}$$

$$= r\,\frac{e^{-2r}}{-2} - \frac{e^{-2r}}{+4}$$

$$= 1.15 \times 10^{-7}\,(2\pi)\left(\frac{re^{-2r}}{-2} - \frac{e^{-2r}}{+4}\right)\Bigg|_0^5$$

$$= 1.15 \times 10^{-7}\,(2\pi)\left[\frac{5 * 4.53 \times 10^{-5}}{-2} - \frac{4.53 \times 10^{-5}}{4} - 0 + \frac{1}{4}\right]$$

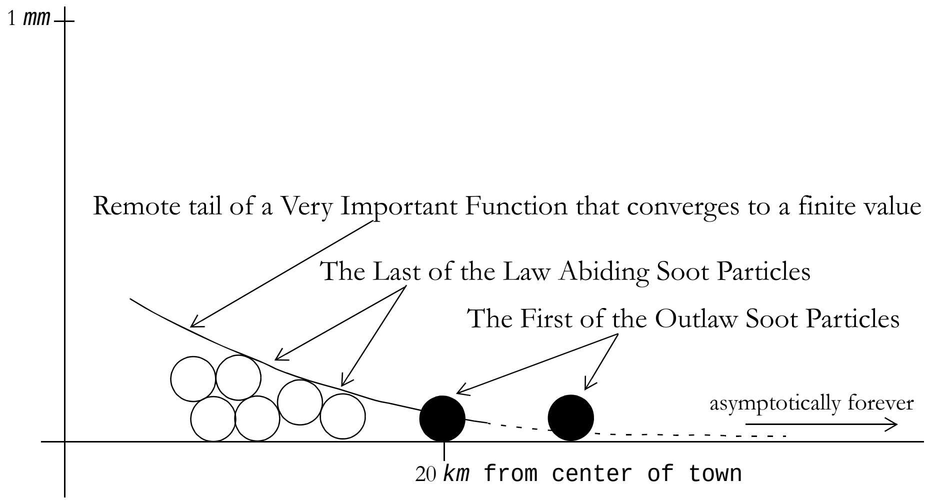
$$= -\,8.18 \times 10^{-11} - 8.18 \times 10^{-12} + 1.806 \times 10^{-7} = 1.805 \times 10^{-7}\ km^3$$

Convert from km^3 to m^3: $\quad 1.805 \times 10^{-7}\ km^3 \left(\frac{1000\ m}{1\ km}\right)^3 = 180.5\ m^3 \approx \boxed{181\ m^3}$

FIGURE 76: The Soot Problem Solved: Integration By Parts

Now we return to Figure 74 to look at the 'zoom' section. Similar to the process depicted in Figure 73, if we zoom in on the far edge of the soot, as depicted in Figure 74b, we find never-ending curvature, within incredibly small y-axis ranges. The relatively tall 'wall' at 0.5 km from the center is only one tenth of a millimeter high. From Figure 75 we know that's the *maximum* height for the soot. So one can scarcely form a mental image of the soot's height at a distance of say 4 or 5 km out from the source. But Figure 74b tells us conclusively that the curve is still falling in that zone, never mind how incredibly small the y-value: it swoops down from one thousandth of a millimeter toward a ten-thousandth of a millimeter.

From the human viewpoint, by the time we reach the 5 km mark, the curtain has fallen on the drama: the curve has completely flattened out and the volume is virtually at `181` m^3, the 'finite' value upon which the integral is clearly converging. (Check: At `10 km` out the integral evaluates to `180.6` m^3 versus `180.5` m^3 at `5 km` out. 'Nothing is happening'.) HOWEVER, from nature's viewpoint plenty is happening in that region, as demonstrated by Figure 74b. Would we see something similar by zooming in at `10 km`? In principle, yes, based on our knowledge of Figure 73, for example. But at some point, we will 'run out of granularity'. In other words, the particles of soot, small though they are, will be incapable of 'honoring' the mathematical model, and at that point the model collapses (in the real world, though not in the imaginary realm).



Eventually there must come a ‘granularity crisis’ such that certain particles of soot fail to honor the on-going function, which in this case was classified as ‘convergent to a finite value’. At this juncture, a constraint in nature has defeated the seemingly ineluctable logic of the function, and the function’s work terminates prematurely. In our picture, we suggest arbitrarily that 20 *km* out is the location of this event that must occur *somewhere* on any such asymptotic curve. From here on, we’ve made the graph a dotted line, indicating failure to model the on-going events. Even in the dotted-line regime, the occasional particle of soot must still exist, but this is ‘outlaw’ soot, still building toward the limit (the aforesaid ‘finite value’), but no longer under the supervision of the function since the function’s asymptotic curve has fallen too low for a soot particle to fit underneath it.

In a conventional treatment, no one worries about this anomaly since the focus is on ‘converging to a finite value’ and one has long since (back at the 5 *km* mark, let’s say) established what that value must be (181 m^3 in this instance) and one has therefore stopped thinking about the function and the fact that *it* goes on forever — or tries to, at least. In the scenario shown here, a presumably Very Important Function has been forsaken not only by the earthling’s short attention span but also by Nature, who perforce goes rogue.

FIGURE 77: Soot’s Journey to Infinity Curtailed

I’ve appended this final example [a] because it is one of my favorite problems, delightful to look at, and [b] to make the point that the kind of patterns shown in Figure **73** can be found in ‘real life’ too; they are not confined to the realm of philosophical discussions about infinity and eternity.

A similar issue arises with **Dead Leaf Density** on page **151**, as the density of an accumulating carpet of dead leaves in the forest *increases* forever, toward a limit. At some point there must be a 'granularity crisis' such that the model will cease to be valid, not through any failure of the model itself, only the physical reality of that particular situation, as it 'runs out of granularity'. I think it is harder to visualize in the leaf problem than in the soot problem, which involves a gradual *decrease* forever, toward zero.

Appendix E: The Naturalness of e Revisited and the Slope of e Demystified, Also Its Entanglement With 1

On the naturalness of e as distinct from a so-called 'natural logarithm' (ln)

On your calculator, there is no doubt a key labeled 'ln'. That's an abbreviation for *logarithmus naturali or log naturalis* — Latin for 'natural logarithm'. I contend that there is *no such thing.* Why? It is the number e, specifically, that may be reasonably characterized as 'natural', not any logarithm attached *to* it.

This may seem like hairsplitting at first, but please bear with me. Indirectly, this will take us to an important and surprising subtopic within calculus that is found in very few other calculus books, if any. (Source: The perspective offered in this appendix is based solely on the wikipedia article 'Natural Logarithm' accessed 07/23/10. I've never seen the slightest hint of it in a calculus textbook.)

First, recall from precalculus that the number e is 'natural' because it keeps turning up in *various* mathematics contexts. Having recognized this special nature of e, *one* of the many things you can do with e is let it be the base of a series of logarithms. And what is a logarithm? That's a fancy name for *exponent*. Nothing more or less. Never forget that. Therefore, it would make no sense to speak of a 'natural exponent' when an exponent is an exponent is an exponent. Neither does it make sense to speak of a 'natural logarithm' since a logarithm *is* an exponent. Rather, what they've done is build a series of humdrum exponents (= humdrum logarithms) *on* the natural number e. Only the base is something special, not the exponents themselves.

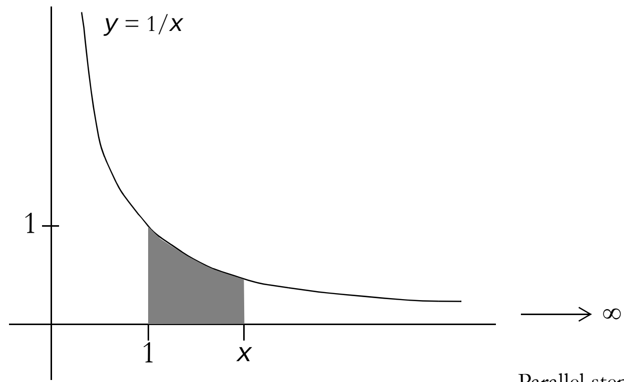
With that bit clarified, we turn to the function $y = 1/x$ to see what the above referenced wikipedia article has to say about it:

> The natural logarithm can be defined...as the area under the curve $y = 1/t$ from 1

to x. The simplicity of this definition, which is matched in many other functions involving the natural logarithm, leads to the term 'natural.'

— wikipedia article 'Natural Logarithm' accessed 07/23/10

My paraphrase: A logarithm of x to base e, the 'natural' number, can be found by measuring the area under the curve $y = 1/x$, from 1 to x, as illustrated in Figure 78.



if $x = 2$, the shaded area = 0.693 square units
if $x = 70$, the shaded area = 4.248 square units
and so forth.

Parallel steps on a calculator:
Press [ln] [2] [=], see 0.693
Press [ln] [7] [0] [=], see 4.248
and so forth.

FIGURE 78: Natural Logarithm defined as an area under y=1/x

"Oh, but I *have* seen that definition in my calculus book," you say. "No you have *not*," I'll wager. Yes, you may have seen the curve $y = 1/x$ with a shaded area marked out, but, very likely, the accompanying text identifies this as the derivative function of $y = \ln(x)$ (as occurs in this volume, for that matter, in **Chapter III: The Fundamental Theorem of Calculus (FTC)**). Or, the caption under the curve might be something to this effect...

$$\text{area of shaded region} = L(x) = \int_1^x \frac{dt}{t}$$

...as in Salas & Hille, p. 342. Those are all true statements, but the thrust of the wikipedia article referenced above is that the crucial insights about e and $y = 1/x$ *preceded calculus* by years or even decades (dating back to Nicholas Mercator in 1668 at least, perhaps back to John Speidell in 1619).

Thus, the kingdom of e can be seen as a kind of self-contained miniaturized precursor or microcosm (my words now) of the larger field of calculus, where e-like relations between a function and its derivative function are extended to encompass, surprisingly, the *entire* mathematical universe.

And by the way, what the heck is a transcendental (irrational) number doing as the BASE (!) of *any* logarithm? Isn't that just plain silly? Yes. And this aspect of the so-called 'natural logarithm' system *is* remarked on at length by Salas & Hille, on p. 340f. (Their proposed alternative approach looks unique, and is well worth reading about.) We now take a second look at the famously special *slope* of *e*.

The Slope of e Demystified

In **Chapter I: Slopes and Functions**, we introduced the slope of *e* on page **31**, using Figure **17** as context. For convenience, we repeat that graphic here as Figure **79**.

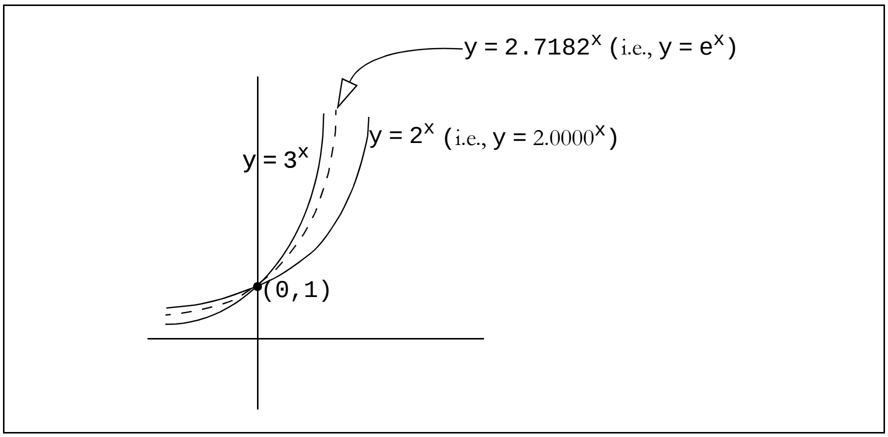


FIGURE 79: The Slope of e in Context (after Stewart p. 422)

As remarked **Chapter I**, 'e^x is its own derivative'. This is noteworthy, yes, but not quasi-mystical, please. Consider the following: For $y = 2^x$ at $x = 0$, the slope is 0.703 (i.e., shallower than 1). For $y = 3^x$ at $x = 0$, the slope is 1.12 (i.e., steeper than 1).

The implication is clear: For *some* number, in-between 2 and 3, there must be one such that the function n^x function has a slope of 1 exactly. It's just a matter of searching in-between the two outer curves depicted in Figure **79** until one finds the sweet spot. That sweet spot happens to be 2.7182.

As proposed in **Appendix G**, there are many circumstances where I believe we would be better off just writing '2.7' (*understood* to be *e*, of course, because of the context) and '3.1' (*understood* to be π, of course, because of the context). My objection to promiscuous use of the symbols π and *e* is twofold: On the one hand, they invest the corresponding numbers with woo-woo mysticism that *maybe* is justified but perhaps not; we just don't know. On the other hand, they imply that the earthling knows 'exactly' what they are, when by definition we obviously have no idea what they are. After all, by our own admission, they are 'irrational'. Adding to the absurdity, the latter is a bit of math-speak that means the opposite of what it says, where π is concerned: The whole bloody problem with π is precisely that it *is* a ratio (i.e., something 'rational' as we redefine that term in passing, by fiat). That's why we can't get our arms around it as a simple number! I shudder to think of the look on a space alien's face, trying to make dignified sense of it all, when really it is all driven by earthbound provincialism.

(Note in passing that a similar mechanism pertains to the logarithmic function: $\log_e 1$ is 0, and the slope at $(1,0)$ is 1.)

Black Jewel or Yet Another Tautology?

Most functions that we are familiar with take a value of *x* and *do* something to it: multiply it by two or raise it to a power, and so on, as suggested by **Figure 12 (The Half-Dozen Ways of Looking at a Function)** on page **23**. A log function does not do something *to* a value of `x`. Rather, it answers a question *about* a value of `x`. Specifically, the `ln` function answers the question: To what power must *e* be raised to *obtain* `x`? That's a very different animal. And in my opinion, it wreaks havoc with Berlinski's valiant attempt at some 'black jewel of calculus' poetry. As a point of reference for our discussion, I've sketched the graph of function y = `ln(x)`, with e on the x-axis and 1 on the y-axis called out, in imitation of Berlinski, p. 280.

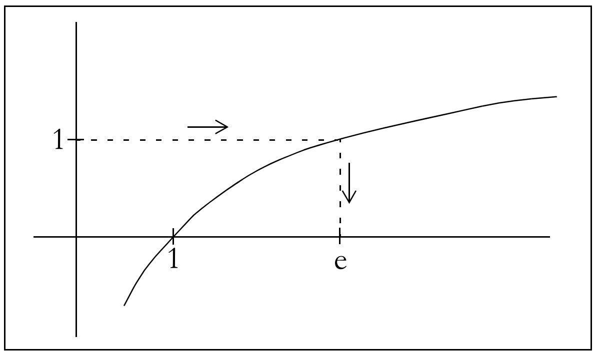


FIGURE 80: An Elegant But Wrongheaded Picture of 'the Black Jewel'

Some of the words accompanying his picture are lyrical and memorable ('...as a stately increasing and continuous function'), but his overall message is just wrong, I think. The general reader won't have time to notice this, but Berlinski (when not in such a moonstruck mood) surely understands the following: Far from being a surprise, the deeply entangled state of 1 and *e* (called out by the orthogonal dotted lines) is inevitable: After all, the whole graph takes e as its foundation, so of course when we reach $e^{x=1}$ in the $e^{x=?}$ series, the expression evaluates inevitably to e. That's all the pretty picture signifies.

Its inside-out nature makes the `ln`(x) function confusing enough already to the first-semester calculus student without this red herring being foisted upon her. This would be a good time to revisit **Figure 13 (White Box View of The Natural Log Function)** on page **24** if one hasn't been there recently. Translating my '2.71' in that figure back to conventional terms, we pose the following question: What exponent of e gives us e? The answer: One, since $e^1 = e$. Is this entanglement of e with 1 *now* a surprise, with quasi-mystical overtones? I should hope not. Try having a second look at the 'black jewel' picture and see if a fog hasn't lifted.

Perhaps I have missed some nuance of the logic, but I find it laughable to suggest, as Berlinski does, that by starting at 1 and traveling back through the function we can 'discover' e on the x-axis:

> ...and *lo*, it is precisely the e of old that now enters into existence, the mysterious transcendental number...not so much defined as *discovered*
>
> — Berlinski, p. 281 (his italics)

Tracing a function from the y-axis to the x-axis is itself a kind of travesty, never mind the particular conclusion proposed in this case.

I shudder to think how many (including myself at one point) have thrilled at the false Nirvana suggested by pages 280-281 in Berlinski, where the caption on the graph reads 'The black jewel of the calculus' no less! What was he smoking that day? (If it was Norbert Wiener writing for the general public, one would suspect a certain well-practiced cynicism, a cavalier attitude about the truth, but with Berlinski it is different. Throughout *A Tour of the Calculus* he seems to fall periodically into a state where he is 'all balls and no brains', his ability to reason temporarily swamped by fantasy-induced hormones. It's really too much.)

Yes, there *is* something about e, I'll admit (*There's Something About Mary*), somewhat like the 'black jewel' quality attributed to it by Berlinski. But the mystery and wonder of e begins *and ends* with $y = 1/x$ (the graph of which Berlinski shows us on p. 279). That *je ne sais quois* has to do with e-ology's role as a *precursor* to calculus, as described earlier in this appendix and found never in the calculus textbooks, which are too busy explaining $y = \ln(x)$ and $y = 1/x$ as a lovely *function / derivative-function* pair (which they *are* of course; they're even my favorites in that role). More specifically, that *je ne sais quois* has to do with the area under the curve of $y = 1/x$, not with the presence of e inside its own function as it were. A more apt phrase would have been 'the black jewel of B.C.' where B.C. stands for the *Before* Calculus era.

Finally, note the close parallel between the overexcitement about 'e^x is its own derivative' (discussed in connection with Figure 79) and Berlinski's overexcitement about the entanglement of 1 and e (which I believe is tautological, a non-event). Elsewhere, in a context that is more deserving of surprise, I've chosen the graph of $\ln(x)$, superimposed on that of Figure 78, as my preferred way to illustrate the

Fundamental Theorem of Calculus (in Figure 20, page 40). So yes, there *is* magic in calculus, at both the macro level (the FTC) and micro level (e.g., in connection with the difference quotient, as noted on page 28), but one should restrain the impulse to shout out its presence, real or imagined, behind every bush.

Appendix F: Symbol List with Annotations

The bulk of this appendix resides in a table where 40-odd symbols, abbreviations and acronyms are listed in a semblance of alphabetical order, each with a definition and annotation.

Outside of the table proper we will begin with an overview of derivative notation. In connection with Figure **2** in the **Prologue**, we sang the praises of prime notation, which is attributed to Joseph Louis Lagrange, 1736-1813 (per Salas & Hille, p. 104n). At some point one needs to realize that y' is only a kind of shorthand abbreviation for the following 'real thing' which is also quite elegant in its own way, coming to us (ultimately) from Leibniz:

$\frac{dy}{dx}$ or $\frac{d}{dx}y$ where $\frac{d}{dx}$ does the job of the prime tick in y'

(For an example of the d/dx form in context, please refer to page **160**.)

Similarly, "the derivative of r with respect to the derivative of t" can be expressed as $\frac{dr}{dt}$ or still more succinctly as $r'(t)$, a convenient form that partakes of the y' concept and marries it to the $y(x)$ format, thus forming a hybrid notation.

In Salas & Hille, p. 5, we are told that the beautiful dx and $\int$ notation was initiated by Leibniz in 1675. True enough. But there is far more to the story than that:

> Those who know something of Leibniz's work know how conscious he was of the suggestive and economical value of a good notation. And the fact that we still use and appreciate Leibniz's ∫ and *d*, *even though our views as to the principles of the calculus are very different from those of Leibniz and his school*, is perhaps the best testimony to the importance of this question of notation... Thus, for considerably more than a century, British mathematicians failed to perceive the great superiority of Leibniz's notation [out of allegiance to Newton]. And thus, while the Swiss mathematicians [the Bernoulli brothers and Euler], the French mathematicians [d'Alembert, Clairaut, Lagrange, Laplace, Legendre, Fourier, Poisson], and many other Continental mathematicians, were rapidly extending knowledge by using the infinitesimal calculus in all branches of pure and applied mathematics, in England comparatively little progress was made. *In fact, it was not until the beginning of the nineteenth century that there was formed, at Cambridge, a Society to introduce and spread the use of Leibniz's notation among British mathematicians*: to establish, as it was said, 'the principles of pure *d*-ism, in opposition to the *dot*-age of the university.'
>
> — Jourdain, p. 57-59, emphasis added

Nor is Newton's stultifying '*dot*-age' the only notation problem ever to have beclouded the mathematics establishment. In my opinion, several of the current practices are equally questionable.

For example, not only is the Fourfold F scheme (introduced as Figure **6** on page **16**) inherently confusing because of its crab canon nature, going simultaneously forward and in retrograde, it is extra confusing because two of its three symbols are overloaded: f serves double duty to denote either the primary function or the derivative function. Also, as discussed on page **29**, f' has been overloaded to mean either the derivative at a *point* or the whole derivative *function*. On page **200** we observe that '=' is triply overloaded to serve sometimes as a verb (*assign a value*), sometimes as an adjective (*equal to*), sometimes as a transformation (x is *transmuted into* y), all dependent on context.

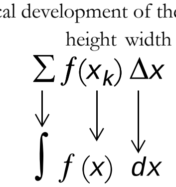
Meanwhile, the Establishment seems to idol-worship certain symbols that are in my opinion overrated (such as π), while seriously denigrating others (such as `lim` and L) that *should* be idol-worshipped by the lowly earthling. For more about symbol overuse and symbol abuse, see **The Dark Side of Notation that is so Terse and Elegant as to be Untouchable** on page **199**f., and **What is an 'equation'? A Very Different Animal in Chemistry, Physics and Math** on page **200**f., and **Appendix D**.

TABLE 9: Mostly Symbols Plus A Few Abbreviations

SYMBOL	DEFINITION	COMMENT
$'$	prime symbol (the derivative of...)	Prime notation (whereby f' is the derivative of f) is attributed to Joseph Louis Lagrange (1736-1813).Without prime notation, calculus homework would be an OK activity. With it, calculus can become a fun, lyrically beautiful pastime.
$\rightarrow$	See Comments	$h \rightarrow 0$ as h approaches zero (Indirectly related issue: triple overloading of '=', page 200.)
CFE	Convergent For Eternity	My own proposed contribution to the jargon. For the rationale, please refer to page 171.
DFE	Divergent For Eternity	My own proposed contribution to the jargon. For the rationale, please refer to page 172.
DNE	Does Not Exist	Abbreviation used to report a negative result when testing for existence of a limit. See page 171.
∇ del alias nabla	See Comment column	∇ is the *differential operator for vectors* (also called the gradient operator in some contexts). The convenient phrase 'curl **F**' is short for the cross product $\nabla \times \mathbf{F}$ where the symbol ∇ represents the upper two rows of the matrix determinant, while **F** represents its third row. For details, please refer to the separate entry for matrix determinant on page 198. (Historical note: The symbol was introduced by Hamilton, per Bressoud, p. 188.)
Δx	delta x: a very small but finite change in x (a perceptible increase)	Compare the entry for dx.
dx	differential of x. An infinitely small change in x (an imperceptible increase)	Compare the entry for Δx. For historical context, see also the entry for $\int$.
Δr	a very small but finite change in the radius	Compare entry for dr.
dr	differential of r. An infinitely small change in the radius	Compare entry for Δr.
δ, ε	delta, epsilon (Greek lower case letters for d & e.)	See **Figure 18 (A Picture of the Limit at Close Range)** on page 35.
∂	'mirror 6', used in lieu of d to indicate a partial derivative	See **Partial Differentiation** on page 90. (Also used as a subscript to indicate 'counterclockwise'.)
X	cross (for forming cross product)	see entry for 'div, curl'

SYMBOL	DEFINITION	COMMENT
div, curl	divergence and curl These are Calculus III topics.	Mnemonic: 'curl cross, div dot' meaning curl equates to the cross product using del notation, and div equates to the dot product using del notation. For an example involving both div and curl in context, see **Green's Theorem — in its Circulation Form AND Divergence Form**, pages **101-117**.
•	symbol for forming a dot product	See entry for 'dot product' in **Appendix G**. See also the entry for 'div, curl'.
e	the natural number, 2.71828	My heterodox opinion: Better to write 2.7 or 2.7182 etc. as appropriate to the context, reserving e itself for special circumstances. Why? 2.7 is humble (but *precise* to n decimals) while e is presumptuous and dishonest, also sometimes quasi-mystical and vague; in a word, *approximate*! (See page 199 and **Appendix E**.)
=	a = 1 (assign 1 to variable a) a = = 1 (variable a equals 1) For use in logic, e.g., if/when a = = 1, then...	For an example of how the '=' versus '= =' issue arises, see page 37 in **Chapter II: Limits**. For a discussion of overloaded symbols generally, see page 200. There exists also a third flavor that doesn't quite fit into the '=' versus '= =' dichotomy: Consider the phrase 'switch to rectangular coordinates' as used in St. Andre, p. 113. He means look at the identities in his conversion table, and use them for substitution. My own informal name for this device is 'sideways identity'.
≤ ≥	inequality signs: less than or equal to greater than or equal to	In calculus, the inequality signs take on a special role, as they provide the basis for a powerful method of defining *bounds of integration*. For an example, see page 216 in **Appendix G**.
f	1. primary function 2. derivative function	See Figure **6** on page 16 and the related diagrams. See also Figure **22** on page 43. (For a definition of 'function' see pages **20-24**.)
f'	1. a derivative at a point 2. a derivative function	See discussion on page 29.
F	antiderivative	See Figure **6** on page 16, also the inverted 'A' at the end of this table.
F′	See Comment column.	This is an unusual alternative symbol for derivative; it is covered indirectly in the entry for del on page 193.
$f(x,y,z)$	scalar function	Compare $\mathbf{F}(x,y,z)$ next.

SYMBOL	DEFINITION	COMMENT
F(x,y,z)	vector function	Handwritten with an arrow in lieu of bolding: $\vec{F}$ Bolded **i** **j** and **k** are likewise vectors.
f_x, f_y	partial derivative	See page 90. (In Schey, p. 4, note 2, the author 'quietly' establishes subscripts on f to denote vector components instead of partial derivatives. This is actually a rather big departure from convention.)
ϕ	A secondary symbol paired with θ when more than one arc variable is needed.	For an example, see **Verify Stokes**, page 143f.
$h \rightarrow 0$	as h approaches zero	Used as subscript to '`lim`', next. (See also '$\longrightarrow$' on page 193.)
`lim` L	limit limit evaluation	For limits generally, see **Chapter II: Limits** and **Appendix D**. For an example of L specifically (as limit evaluation), see page 34.
LNR	Limit Never Reached	My own proposed contribution to the jargon, in lieu of 'limit'. Absent this reminder, one assumes the whole point is to 'flea-hop' *onto* the limit at one's earliest convenience, thus making a mockery of the whole business.
parameters *replaced* on an ad hoc basis by numeric values	Example: The use of a nice pleasant-looking number such as '1' or '100' as if it were a parameter, when one ought to be writing on the white board the name of an actual parameter such as 'Maximum_x' (either mindlessly because the latter has not yet even been conceived, or with intent because it has been conceived but is deemed an ugly baby, too hard to look at).	I'd wager that most instructors who engage in this practice are not even aware of what they are doing; it is simply 'the way things are done'. That does not make the practice any less deplorable. It is, after all, a reversal of the idea that lies at the very foundation of algebra: the substitution of letters for numbers! Because they are the *sine qua non* of many functions, we should always use the parameters themselves, even if they strike us as bothersome or 'ugly'. For an example of numbers used as ersatz parameters, note the role of 1 and –1 in Hughes-Hallett p. 397 #9.

SYMBOL	DEFINITION	COMMENT
π	pi	My heterodox opinion: Better to write 3.14 or 3.1 etc. as appropriate to the context, reserving π itself for special circumstances, e.g., for use on the x-axis when plotting trig functions (as in Figure **9** on page 19, e.g.) Why? Because 3.1 is humble (but *precise* to n decimals) while π presumptuous & dishonest, also sometimes quasi-mystical and vague; in a word, *approximate*! (See page 199.) Historical note: π as a designation for the ratio of circumference to diameter was introduced in 1706 by William Jones, "probably after the initial letter of the Greek περιφερεια, 'periphery' ." Thirty years hence, this usage was given a boost when Leonhard Euler adopted it in lieu of *p* (Gullberg, p. 85-86).
$\int$	integral sign	Leibniz's stylized 'S', chosen because the integral is a limit of *sums*. E.g., the area under a curve can be estimated as a collection of rectangles, i.e., a Riemann *sum*. Historical development of the summation symbols:  — after Priestley, pp. 253-254 Compare $\Delta y * \Delta x$ on page 63. See also page **218**.
$\int$ stuff	'stuff' is the integrand to be integrated	The stuff to the right of the stylized 'S' is said to be 'under the integral sign'.
$\int Q = P$	'The antiderivative of Q is P' or 'The derivative of P is Q' or 'P is the antiderivative of Q'	Why P & Q? Why not follow the convention that uses f' and f or f and F? In order to highlight the (uncomfortable) bridge between the symbolic language and English language, I am using P & Q as a kind of metalanguage here. For another angle on this, see 'English language P/Q trap' in Table 10 on page 226.
$\int f(x)\ dx$	indefinite integral of $f(x)$ with respect to x	See the entry in **Appendix G** for 'indefinite integral'.
$\int_0^\infty f(x)\ dx$	definite integral of $f(x)$ with respect to x, on the interval $x = 0$ to $x = \infty$	

SYMBOL	DEFINITION	COMMENT
$\int_C P\,dx + Q\,dy$	$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy$	Elsewhere, I've mention the problem of 'overloaded symbols' (page 200). Here we have instead the problem of a sphynx-like abbreviation. Understandably, the establishment is in love with the handy abbreviation in column 1, but rarely do they explain that it is *only* a slapdash abbreviation, for what appears in column 2. Similarly, the expression $\int \mathbf{F} \bullet d\mathbf{r}$ is actually just the 'skin of an onion' with many layers. (For a glimpse beneath the surface, see 'Line Integral of a Vector Field' in **Table 7** on page **122**.)
$\oint$	loop integral	alias contour integral $\int_C$ or $\oint_C$
$\Big[\text{ stuff }\Big]_5^{20}$	'stuff' is being integrated over the interval x=5 to x=20.	Large square brackets, with superscript and subscript attached to the right bracket. Typically this occurs toward the end of an integration calculation. It double as a wonderfully succinct notation convention whereby an author can excerpt just this sliver of a longish integration computation to imply the whole (as in Priestly, p. 251, for instance).
$\text{stuff}\Big\|_1^{\infty}$	'stuff' is being integrated over the interval x=1 to x=∞.	A variation on the above notation convention, still more elegant and succinct. For an example in context, see page 73.
[1, ∞]	the interval 1 to ∞	Horizontal notation for expressing 'the interval 1 to ∞' (generic notation, not part of integration notation; compare square brackets with superscript and subscript above)
$\|x\|$	absolute value of x	alias ABS
$\|\mathbf{r}'(t)\|$	the *length* of an arc (or 'norm')	Typographically, these vertical bars are no different from the ones that mean 'absolute value', listed immediately above. Their meaning is determined by context. (Since 'norm' is a synonym for vector length, these vertical bars may also be referred to as 'norm signs', e.g., in Spivak, pp. 1 and 16.)

SYMBOL	DEFINITION	COMMENT
$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial}{\partial x} \\ P & Q & R \end{vmatrix}$	matrix determinant of order three	Read this entry in tandem with the entry for del: ∇. In curl notation, the symbol ∇ represents the upper two rows of the determinant. Those two rows are static (i.e., the *same* for all problems). Meanwhile, the third row, *PQR*, is represented by **F**. The third row is the only part of the determinant that changes from problem to problem, so 'it's not as bad as it looks'. (See also **Implicit y, Implicit z, Implicit w** on page 134.)
$\sqrt{3}$	square root of 3	My heterodox suggestion: Whenever it is practical, write '1.73' in preference to $\sqrt{3}$. The rationale is given in **The Dark Side of Notation that is so Terse and Elegant as to be Untouchable** on page 199. (Also, in **Appendix C** note the unpleasant consequences of being locked into $\sqrt{3}$ and $\sqrt{2}$. Thus, my advocacy for moving away from the 'exact value' orthodoxy has both theoretical and practical motivations.)
$\forall$	shorthand symbol for antiderivative?	This is my own half-serious proposal for a new symbol: Borrow the inverted A from logic (where it means 'for every'), and let it stand for 'antiderivative', in case the latter takes too long to write down during a lecture, and/or the symbol 'F' strikes one as too subtle by half.
∞	infinity	It comes with fine print: 'This is not a number'. See page 216.

The Dark Side of Notation that is so Terse and Elegant as to be Untouchable

In a context such as **Figure 9 (Two Notions of the Sine Wave)** on page **19**, I'm as happy as anyone to partake of the elegance and convenience of the symbol π, and will gladly play along. Likewise *e* in a context such as the partial differentiation example on page **91**. But there is an aspect of these symbols that I find unsettling. Intentionally or not, a symbol can become a kind of mask for the high priests of mathematics to hide behind, sealing off certain avenues of discussion.

For example, where the establishment sets up a contrast of...

'exact'	'approximate'
π	3.1416

...in which the words 'exact' and 'approximate' are elevated to the status of technical terms beyond reproach, I would propose characterizations that turn the conventional ones their heads:

presumptuous-yet-*vague*	humble-yet-*precise*
π	3.1416

I reject the whole paradigm, especially where it involves numbers that hide behind a Greek letter or under a surd ($\sqrt{2}$, $\sqrt{3}$ ... (Admittedly, *e* is a special case; it is discussed separately in **Appendix E**.)

Look at it this way: In a word problem involving rainfall, you use '67% chance' as an illustrative value. The teacher ignores the overall logic of the problem and marks the whole thing wrong, having zeroed in with a fury on your '67% chance of rainfall'. His explanation, scribbled in the margin: "67% is only approximate. You should always cite ρ instead because ρ is the precise chance of rainfall on any given day, to an infinite number of decimals." Wouldn't you feel the instructor's love affair with the Greek letter *rho* and his use of the English language were both rather...peculiar? (As if to slyly undermine this cockeyed usage of ours, Indian English revels in the expression 'exactly and approximately' as the all-purpose qualifier for monetary terms, temporal terms, and others.)

Related discussion: **Round-tripping 45 Degrees Through a Calculator in Mode Radians** on page **166**.

For an example involving both *e* and π, consider the soot integral on page 176:

$$0.000000115e^{-2r}\ km\ 2\,\pi\ r$$

My recommendation would be that we rewrite it as follows...

$$0.000000115(2.7)^{-2r}\ km\ 2\ (3.1)\ r$$

...the theory being that '2.7' and '3.1' are perfectly recognizable *as* the Natural Number and the Ratio of a Circumference to its Diameter precisely because they are ubiquitous; and to express them in simple rustic numerals is more becoming of an ignorant biped earth-dweller than to use ivory tower symbols 'with attitude'.[36]

What is an 'equation'? A Very Different Animal in Chemistry, Physics and Math

A chemical equation has this general appearance:

$$CH_4 + 2O_2 \rightarrow CO_2 + 2H_2O$$

In particle physics, a decay formula has this general appearance:

$$n \rightarrow p + e^- + \bar{\nu}_e$$

In either case, one *could* substitute '=' for '→', by the rationale that 'this is an equation [of such-and-such special type]'. But why do such a silly thing? Why indeed!

Yet in mathematics and physics (excluding particle physics) the distinction is ignored: Here, entities that *should* be joined by an arrow are in fact joined by an equals sign instead. The equals sign is thus 'overloaded'. And the equals sign is overloaded another way as well: 'A = B' means either 'LET A equal B' or '[IF] A is equivalent to B', dependent on context. In the former case we are assigning a value *to* A. In the latter case we are setting up a test: *Are* A and B *already* equivalent without me even touching them? That's the implied question. These two operations live on opposite sides of the logic universe and deserve separate symbols: The first is algebraic and it works like a verb; the second is Boolean and it works like an adjective. In software engineering, there is no such overloading of the symbol. Rather, the two cases are parsed out this way:

'A = B' means *assign* the value B to A.
'A = = B' means A is *already* equal to B.

In summary, in the math-physics tradition, the symbol '=' is overloaded with at least the following three definitions...

1. = (shall have *assigned* to it...)
2. = = (is *already* equivalent to)
3. → (is transmogrified to *become*)

...the first of which involves a right-to-left operation (like a ← 1), the second of which is static, and third of which involves a left-to-right process. That's quite a burden for one tiny symbol to carry.

I think I know deliberate terseness and elegance when I see it. (Exposure to Classical Chinese, which was a big part of my training as a sinologist during the period 1957-1975, gives one an appreciation for terseness.) But '=' in mathematics has the whiff of something else: ossification and insularity. Insularity from computer science, where '= =' waits patiently, as it were, ready to solve the second part of the overloading problem. Insularity from chemistry, where '→' could solve the third part of the overloading problem. (Actually, the right arrow *is* part of the mathematics vocabulary, as in $\lim_{h \to 0}$ or f: $\mathbf{R}^n \to \mathbf{R}^m$ meaning f takes $\mathbf{R}^n$ into $\mathbf{R}^m$ (Spivak, p. 11). So the real question is why the arrow isn't used more extensively, to include situations such as the one illustrated in Figure **43** on page **77**.)

When the going gets tough in a math or physics class, often you are running up against something that is inherently difficult, but realize now that in many cases the trouble arises from the notational practices summarized above (which are compounded by a cavalier attitude in math and physics whereby the distinction between a value and a parameter is often annihilated in a wrongheaded vote for convenience; see page **195**).

Appendix G: Glossary of Jargon from Antiderivative to Wonk

In alphabetical order, I list out the main items of jargon that one will encounter in elementary calculus — some of them wonderfully useful, some of them rather aggravating.

antiderivative (and anti-work)

Suppose I go from home to work then back again, and on the return trip I say, "I'm going to my anti-work." That would be analogous to the calculus lingo we are trying to illuminate here. My *home* hasn't changed, only my *name* for it, coming and going. Or, to put it in terms of Figure 6 on page 16, when the tree is inferred from its shadow, the tree is called an antiderivative. But it's still the same tree as ever. Thus we find that this notion of 'antiderivative' might carry less weight and complexity than appears at first. The antiderivative F is simply the *function* f now approached *from* the viewpoint of the derivative (the shadow). Meanwhile, nothing has changed — neither the tree in Figure 6 nor '$y = \ln(x)$' in Figure 22 has been altered by any of this talk.

Can we then dispose of the term 'antiderivative' as a mere tribal quirk handed down by the pedagogical tradition? Not quite. When it comes to the question of '*the* antiderivative of the function f' (incorrect) versus '*an* antiderivative of the function f' (correct), we see that my attempted analogy with 'anti-work' breaks down, suggesting that some important subtlety has been neglected thus far. For the rest of the story, please refer to **Constant Rule and '+C'** on pages 78-80, where I document another flavor of borderline absurdity (which is the ultimate rationale for the existence of the term 'antiderivative').

antidifferentiation

When thinking about antiderivatives (*q.v.*), an author may sometimes write 'antidifferentiate' in lieu of 'integrate'; e.g., in Hughes-Hallett *et al.*, p. 312.

approximate

In math-speak, the words 'approximate' and 'exact' are elevated to the status of technical terms that wind up meaning precisely the opposite of their commonsense definitions. The semantic neurosis aside, I have objections on theoretical grounds; see **The Dark Side of Notation that is so Terse and Elegant as to be Untouchable** on page **199**.

asymptote

The term asymptote is sometimes used as synonym for 'limit', e.g., in Hughes-Hallett *et al.*, p. 553. See discussion on page **170**.

cancellation property

There are two cancellation properties that come up in elementary calculus:
1. The cancellation property of the FTC, whereby $\int f' = f$
2. The cancellation property of e and $\ln$, whereby $e^{\ln(x)} = x$.

In words, "If you take the derivative of a function then integrate the derivative, you get the function back. The two operations cancel each other out." (Regarding f' f notation versus f F notation, see page **46**.) Or, in generic terms, "If you perform *both* operation J *and* operation K, its opposite, against x, then J and K cancel each other out, and you get x back, all by itself." These cancellation properties are sometimes very handy, and always fun to write and say (especially 'e to the Ellen x equals x'). The FTC cancellation property can be seen in action on page **99**; it provides one of the crucial steps in the derivation of Integration By Parts from the Product Rule. An example of the second cancellation property listed can be seen on page **153**, where it combines powerfully with exponentiation (q.v.).

capping surface

The concept of a *capping surface* provides the basis for one of the most dramatic demonstrations of the FTC. Curiously, textbooks seem never to mention it by name. Rather, one might stumble on the term while reading Schey's classic:

> Suppose we have a closed curve *C*...and imagine that it is made of wire. Now let us suppose we attach an elastic membrane to the wire...This membrane is a capping surface of the curve *C*. Any other surface which can be formed by stretching the membrane is also a capping surface...the plane enclosed by a circle [the place where Green's Theorem operates]...a hemisphere with the circle as its rim [as in **Verify Stokes** on pages **143**-**147** above]...the curved surface of a dunce cap...the upper and lateral surfaces of a tuna fish can.
>
> – H.M. Schey, pp. 93-94, in his prelude to Stokes' Theorem

The concept is important because it immediately clarifies the difference between Stokes' Theorem and a Surface Integral (area of a curved surface as it relates to the projected 'checkerboard' beneath it) and because it adds a sort of exclamation mark to Green's Theorem, thus raising the ante once again, toward ever greater heights (for context, see Figure **63** on page **127**).

chain it (as in "You forgot to chain it")

The verb 'to chain' is shorthand for the following chunk of verbiage:

> The act of writing 'd/dx[stuff]' where 'stuff' is a copy of the inner part of a compound function, and 'd/dx' earmarks said copy of the inner function as an item you intend to differentiate in a subsequent step

The way I've stated it here, I admit it sounds crazy, but if you read this page in tandem with the examples under **Chain Rule: Single Variable** on page **83**, I believe the fog will clear. Long story short, in calculus classes one often hears or sees the phrase '*you forgot to chain it*', and the act described above is what the student neglected to do. So this is an important term, amounting almost to a little subtopic in Calculus I, albeit awkwardly explained by me.

concave up / concave down

See **Chapter IV: Curves!**, especially Figure **25**, page **48**.

convergent/divergent

See the entry for 'divergent'.

converging/diverging

See the entry for 'divergent'.

* convex

No such thing in calculus! Instead, one speaks always of 'concave down' (which looks to me like 'convex' but maybe I'm crazy). I have a theory about this odd gap in the math lexicon: Once upon a time, a calculus teacher grew frustrated with his/her students' inability to keep the terms 'concave' and 'convex' straight in their heads. (They are a bit confusing.) In a moment of sardonic wit, the teacher said, "OK, let's just call the curves 'up' and 'down' — no, better yet, let's call them 'concave up' and 'concave down'. How about that? Heh, heh, heh. Then you needn't be confused *ever again*." And it stuck. So, blame the students for this one? Maybe, but my story is too weak a reed to support that conclusion. The truth is probably even stranger than the story I've concocted. But one thing is for sure: calculus has nothing convex in it. (For more about this, see **Figure 29 (Curves 101)** on page **51**.)

curl

See 'divergence and curl'

cusp

See **Figure 19 (Limits, Continuity, Differentiability)** on page **37** and the discussion of cycloids on page **56**.

del ∇

The differential operator for vectors (also called *nabla*, the Greek for an Egyptian or Hebrew harp).

If the FTC were *routinely* notated this way $\int f'(x)dx = f(b) - f(a)$ (a reasonable practice adopted by Protter and Morrey, on p. 445, e.g.), then it would be helpful to say "∇f is the higher dimensional analog of $f'(x)$", as in St. Andre, p. 210. But since the FTC is usually written this way $\int f(x)dx = F(b) - F(a)$, using *anti*derivative notation instead, there exists a continual kind of dissonance that interferes with the desired analogy.

(On page **238**, we see Stewart opting for an exotic looking F'/F combo in row 1 of his summary table, presumably to bring out the analogy with ∇f and f in row 2. Nice as far as it goes? But then his row 1 is at odds with every other statement of the FTC that one will ever encounter elsewhere. No easy way out of such a notational tangle. Moral of the story: Notation matters!)

derivative (at a point)

To avoid getting lost in the details, it is good to keep in mind the following definition:

> The Derivative: A Fancy Calculus Word for Slope and Rate
> — heading in Ryan, p. 59

If that sounds too glib or giddy for the your needs, try the section entitled **The Difference Quotient (alias 'limit definition of the derivative')** which begins on page **27** above.

derivative function

The concept of a (whole) derivative function is best understood by contrasting it with the notion of a derivative at a point. While there is a sharp distinction between the two concepts themselves, they are easily blurred by the notation convention (f'). See pages **27**-**30** above; compare Figures **15** and **16** on pages **28**-**29**.

differential (as a noun)

See the entry in **Appendix F: Symbol List with Annotations** for 'dr'.

differential calculus

The subject taught in Calculus I. Compare 'integral calculus'.

differential equation

If an equation contains an unknown function and one or more of its derivatives, it is a differential equation (paraphrase of Stewart, p. 618). How can an equation both 'contain' something and yet 'not know' what it is? By analogy with 'x' being *in* an equation yet being simultaneously '*the unknown*'. Same idea, cranked up a notch: now it's a whole function that we are solving for, not just a missing value. Where do these differential equations come from? They are not exercises made up by perverse educators along the lines of 'Anything that doesn't kill you will make you stronger'. Rather, they are given to us — in droves — by nature. They are the result of observing a change in nature, then posing the question, 'What function is driving that change?' so that one might graph the function and use it to predict the future.

Typically differential equations are introduced toward the end of Calculus II, then much of Calculus IV is devoted to them. In this book, there is only one example of such, presented and analyzed on pages **151**-**156**.

differentiate

Please refer to page **27**f in **Chapter I** and page **75** in **Chapter VI**. See also the entry below for **"Let's differentiate [stuff]" meaning "Let's flag [stuff] for differentiation someday"**..

divergence and curl (div and curl)

These are Calculus III topics, which we cover to some extent in **Chapter VII**.

divergence theorem

Another name for Gauss's Theorem. This is discussed in **Chapter VII**.

The practice is to call it the Divergence Theorem; I prefer to use the other, less frequently encountered name for it: Gauss's Theorem

divergent

Since 'divergent' and 'convergent' are the two opposite flavors of an improper integral, one may imagine that 'divergent' indicates the kind that keeps growing forever toward infinity. *Often* this is true, but one should be aware of the following linguistic trap: In calculus, 'divergent' is actually a technical term that simply means '*not* convergent'. In being not convergent, the improper integral may do something other than grow forever. For instance, it might only pingpong mindlessly between +1 and –1. Thus, unlike 'convergent', the term 'divergent' is not a descriptive term. Rather, it is a tricky and deceptive synonym for 'limit does not exist' (DNE) which may sometimes *also* be descriptive.

Note that diver*gence*' connotes The Divergence Theorem (alias Gauss's Theorem), listed separately above. Therefore, one speaks specifically of an improper integral that is 'convergent' or 'divergent' and does not speak in that context of 'convergence' and 'divergence' (even though it would be natural to do so). Presumably, this unwritten rule is for the sake of preventing a 'collision' with the name of The Divergence Theorem.

Indeed if one goes back to an earlier archeological layer to look at, say, *Calculus*, Fourth Edition in the *Schaum's Outlines* series, she finds the curious assertion on page 396 that the following is The Divergence Theorem: "If $\lim_{n\to+\infty} s_n$ does not exist or $\lim_{n\to+\infty} s_n \neq 0$, then Σs_n diverges.” Meanwhile, its index includes no entries at all pertaining to Green’s Theorem, Stokes’ Theorem or Gauss’s Theorem (aka The Divergence Theorem).

diverging

See converging/diverging

dot product

The dot product of vectors $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ and $\mathbf{v} = c\mathbf{i} + d\mathbf{j}$ is denoted $\mathbf{u} \bullet \mathbf{v}$,
and is defined to be the real number $ac + bd$. Thus: $\mathbf{u} \bullet \mathbf{v} = ac + bd$.
For an example in context, see page 144 above.

evaluate

Here we move beyond the mundane annoyance of jargon into borderline neurosis (or just plain laziness?):

> [In this first set of problems] Use Stokes’ Theorem to evaluate $\iint_S \text{curl } \mathbf{F} \bullet d\mathbf{S}$
> — Stewart, p. 1143
>
> [In this second set of problems] Use Stokes’ Theorem to evaluate $\int_C \mathbf{F} \bullet d\mathbf{r}$
> — Stewart p. 1143

The two statements above, occurring in a volume that is otherwise one of the finest calculus textbooks on the market, are close to gibberish. Eventually, one can infer that they are trying to express something that is essentially the *opposite* of what they appear to say. Here is what they actually mean:

> In this first set of problems, *pretend* that you wish to evaluate the *left* side of Stokes’ Theorem ($\iint \text{curl } \mathbf{F} \bullet d\mathbf{S}$), and instead evaluate the *right* side of the theorem, namely $\int \mathbf{F} \bullet d\mathbf{r}$, to get your answer (since by definition the left side and right side are set equal to one another, so it doesn’t matter which side you use).
>
> Conversely,
> in this second set of problems, *pretend* that you wish to evaluate the *right* side of Stokes’ Theorem ($\int \mathbf{F} \bullet d\mathbf{r}$), and instead evaluate the *left* side of the theorem, namely $\iint \text{curl } \mathbf{F} \bullet d\mathbf{S}$, to get your answer (since by definition the left side and right side are set equal to one another, so it doesn’t matter which side you use).

This quirky use of the word ‘evaluate’ is related to the discussion of Table 7 on page 122. See also the related entry on page 219 below.

In case one begins to doubt his/her sanity and wishes to confirm that ‘evaluate’ actually *means* 'evaluate', one may consult Woods, page 5: "Verify Green's Theorem by evaluating both sides of the equations [for Flux and for Circulation, each in turn]." It's a similar context, with the word 'evaluate' now used in a pleasantly rational (nonpsychotic) manner.

exact

In math-speak, the words 'approximate' and 'exact' are elevated to the status of technical terms that wind up meaning precisely the opposite of their commonsense definitions. The semantic neurosis aside, I have objections on theoretical grounds; see **The Dark Side of Notation that is so Terse and Elegant as to be Untouchable** on page **199**.

exponentiation

Sometimes you take an equation, say $p = q$ as an arbitrary example, and turn it into $e^p = e^q$ instead. You have thus 'exponentiated' the equation. To see *why* you would wish to do this funny thing to an equation, dressing it up as powers of e, you must look at the device in context. See, for example, the situation on page **153**, where exponentiation is combined powerfully with the cancellation property (q.v.) for e and `ln`.

finite

An integral that converges on a specific number is called 'finite'. Really, in this context, 'finite' just means *not* 'infinite'. The terminology in this whole area of calculus is (another) train wreck in my opinion. See **Appendix D**.

FTC

The Fundamental Theorem of Calculus:

$$\int_a^b f(x)\,dx = \mathrm{F}(b) - \mathrm{F}(a)$$

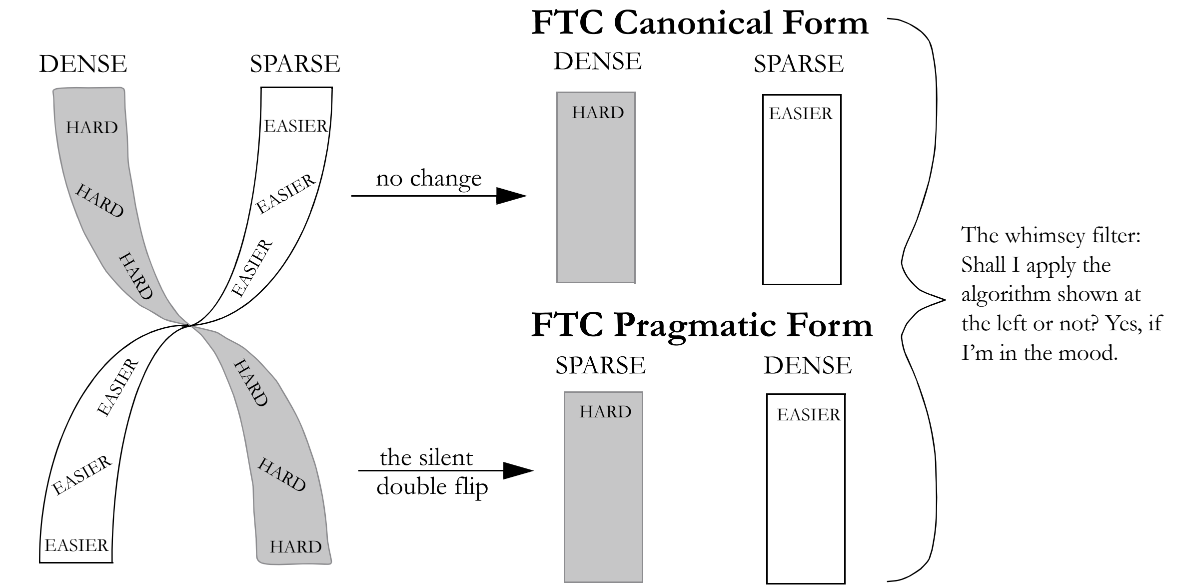
Please refer to **Chapter III**.

FTC Canonical Form versus FTC Pragmatic Form

The two terms in this heading ('FTC Canonical Form' and 'FTC Pragmatic Form') are my own. One might apologize for adding to the lexicon this way, except that the distinction *needs* to be articulated, and it lacks a name at present, so far as I can tell.

When referring to *the* FTC of elementary calculus, the vast majority of authors will show it with the integral on the left and the simple part on the right, as in the '**FTC**' entry immediately above. Given that context, it is natural for the student to develop a notion that this is the theorem's *canonical* form. (And in fact it is. Only rarely, as in Apostol, p. 202, do we see it turned around, with its antiderivative caboose coming ahead of the engine, so to speak.) Eventually, the student sees *the* FTC placed in juxtaposition with certain FTC *variants* (as sampled in Table **6** on page **120** for instance), and now it is reasonable for her to assume that the canonical form carries also a message about dimensions and about one's expected exploitation of the theorem: one uses the easier, n-dimensional, *right* side of the equation to do an end run around something harder and $n+1$-dimensional on the *left* side, or so it seems for a while. Or, at a higher level of abstraction, one may propose an overarching principle: "Notice that in each case we have an integral of a 'derivative' over a region on the *left side*, and the *right side* involves the values of the original function only on the boundary of the region"; Stewart, p. 1152, my italics. Meanwhile, *further* reinforcing the idea that theorems and functions have a distinct 'left side' and 'right side', we have the definition of a function as given by Gullberg, which includes this assertion: "It is standard practice to write the **dependent variable** on the left-hand side of the equality sign of [the function]" (quoted with context on page **21** already).

However, for theorems specific to Calculus III, it seems that authors and instructors silently abandon all the above principles, implicit or explicit. Suddenly everyone is quite casual about the left side versus the right side. It is as if your authors and instructors had suddenly conspired, for some odd reason, to start writing functions *either* as $f(x) = x^2 + 3$ *or* as $x^2 + 3 = f(x)$, all on the flip of a coin, with cool and sphinx-like aplomb. Why? After so many high-flown theories and abstractions, the explanation for this practice is anticlimactic: Floating invisibly above all the other unwritten rules and principles, it turns out that there is a Top Rule: The Top Rule says, Hard on the left, Easier on the Right. See Figures **81** and **82**.



An inherently simple pattern in calculus itself (DENSE/SPARSE) is pushed by *Homo sapiens* through a kind of 'gene splice' dance, which results in the appearance of FTC Pragmatic Form as a tacit alternative to FTC Canonical Form, used sporadically, subject to the whimsy filter: 'Only when I happen to feel like it'.

FIGURE 81: How a Silent 'Gene Splice' Leads to FTC Pragmatic Form

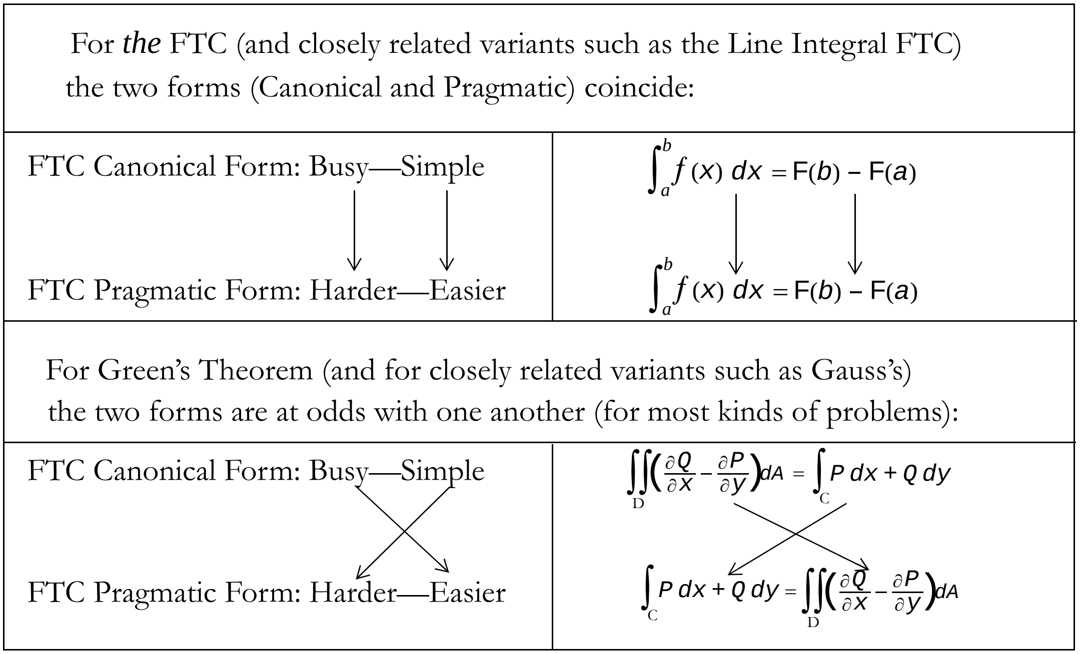


FIGURE 82: FTC Canonical Form and FTC Pragmatic Form

To see how this plays out in the larger picture, please refer to Table **7** on page **122**. See also Figure **63** on page **127**.

A closely related issue: The quasi-neurotic (mis)use of the seemingly straightforward term *evaluate*; see page **209**. The situation turns almost comical when we find someone speaking of Green's Theorem being used 'in the reverse direction' (Stewart, p. 1105) meaning that one might, exceptionally, evaluate $\int$, not $\iint$, to solve a particular problem, e.g., when using Green's area theorem. Thus, the canonical form is deemed 'backward' for the nonce, since pragmatic form is expected.

function

Defined on pages **20**-**24**.

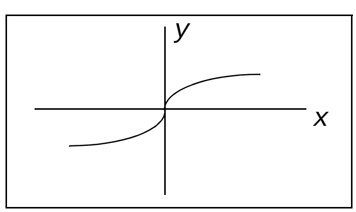
function notation

The rudiments of function notation are given on page **23**. Sometimes a function is defined in two pieces, as follows:

$$f(x) = \begin{cases} 1, x \text{ rational} \\ 0, x \text{ irrational} \end{cases}]$$

As an example, we've shown the Drichlet function above. Here is what the notation means: If x is rational, then y is 1; if x is irrational, then y is 0. Thus, a simple concept has been turned into a rune by the notation which first states a result *then* gives the condition required to arrive at that result! (Shades of the proverbial farmer who tells you to make a left turn five miles before you get to the barn.) The same ethos appears in the peculiar English of engineering majors, which invariably following this syntactic pattern: "See Example 27.1 on page 834 and Section 27.3 on page 841 in Raymond A. Serway and John W. Jewett, Jr., *Physics for Scientists and Engineers* [6th Edition, 2004] if you are interested in the obscure side-issue of 'electron drift velocity'." Whereas, if one were thinking logically or if one had the slightest consideration for the reader, one would structure such a sentence the other way around: 'If interested in A, consult B which is found in JKL and PQR as summarized in XYZ'. This way, in the likely event that topic A is of no interest to the reader, she may bail out of the sentence almost immediately instead of being dragged through to the bitter end to see what it's about.

Here is another way of (partially) defining a function: The practice of writing '$f(x) > 0$ for all $x > 0$'. In the database world, one is interested to know if a value is zero or non-zero because in many situations that's the difference between *nothing* and *something* (to be processed). Next, if the item is non-zero, one is interested to know if it is positive or negative, so that one can decide *how* to process it. (I've omitted a third flavor, *null*, which would add nothing to the current discussion.) On first encountering the expression above (with its two instances of '> 0'), someone coming from the world of computers is inclined to think that here, too, the interest might be in discriminating zero from something non-zero. But really what the phrase means is this: 'If x is positive, y is positive', a condition that would hold for the following graph, for instance.



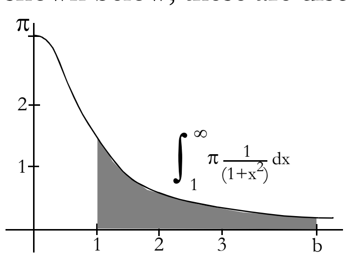
So, once again, we see a case where simple logic has been tortured into a kind of pretzelled rune from outer space. Why? (As an erstwhile sinologist, I am a connoisseur of tersely pregnant language, epitomized by Classical Chinese, but this is something else. The ethos driving the syntax documented in this section is not one of economy or elegance; rather, it smacks of passive-agressive obnoxiousness. And math teachers wonder why the general public runs screaming from their subject? If one could fight his/her compulsion to say 'B contradesignates A' where everything that needs to be said is covered already by 'A is B'...)

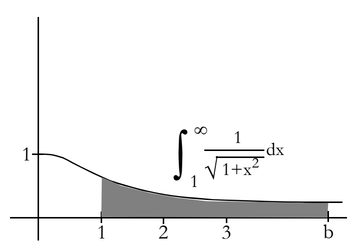
gradient

Not a topic in this book, but in case one is looking for a definition, this is the best I've seen: "The partial derivatives $f_x(x, y)$ and $f_y(x, y)$ may be thought of as the derivatives in the x direction and y direction, respectively. This section defines the 'directional' derivative for an arbitrary direction $\mathbf{u} = \langle a, b \rangle$ in the xy-plane. $f_x(x, y)$ and $f_y(x, y)$ will be special cases where the x direction is specified by $\mathbf{i} = \langle 1, 0 \rangle$ and the y direction by $\mathbf{j} = \langle 0, 1 \rangle$. The vector that points in the direction where the directional derivative is the largest is called the gradient." St. Andre, p. 151.

improper integral

Most of the integrals labeled 'improper' are really about time, not about stuff. In lieu of 'improper', I propose that they be called 'eternal'. Two examples of an 'improper integral' are shown below; these are discussed at length in **Appendix D**.





indefinite integral

As often happens in the graceless tortured language of Mathlish (a dialect of English), we find a word being coerced to convey something at odds with its dictionary definition.

An ‘indefinite integral’ is actually a generic, nonspecific integral. There is nothing even slightly ‘indefinite’ about it:

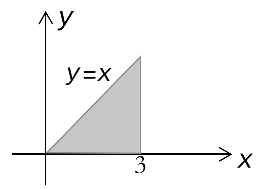
$$\int f(x)\, dx$$

(George has a doozy of a urinary track infection. Who knows where he got *that*? Anyway, it’s a ***definite*** condition, not ‘indefinite’. But it happens in this case to be ***nonspecific*** or ***generic***: the blood tests have not matched his ailment to any specific pathogen.)

inequality

From precalculus, I recall pages and pages of inequalities, which possess *some* intrinsic interest, yes, but in retrospect it is astonishing that the precalculus curriculum gives not the slightest clue of how these inequalities will actually be *used* in calculus. Which is quite interesting.

Here is an illustrative example: Assume a certain vector function is to be evaluated by means of a double integral over the triangular region shown at the right. As a puzzle for the human eye, this is trivial, but translation of the picture into mathematical terms is surprisingly tricky. At least I find it difficult, as the language of inequalities has a terseness about it reminiscent of Classical Chinese:



$$y \le x \le 3,\ 0 \le y \le 3$$

The inequalities above represent the ‘heavy lifting’ part of the translation, following which the double integral almost evaluates itself, as it were, painlessly:.

$$\boxed{\int_0^3 \int_y^3 dx\, dy}$$

The example is after Wood, p. 4, highly abbreviated for this context.

infinity

It comes with fine print: ‘This is not a number’. Shades of Michel Foucault’s picture of a pipe with caption, ‘This is not a pipe’.

My heterodox opinion: Avoid the whole issue by saying ‘for eternity’ not ‘to infinity’; for more about this issue, see **Appendix D**.

Why do I consider infinity suspect? For one thing, it is easily derived from zero. Here’s how: First, we observe (in Nature) that the derivative of a constant, C, is zero. Thus, $C' = 0$, a true statement. (See **Constant Rule and ‘+C’** on page **78**.) Next, we sidestep the conundrum $\int 0$ by writing $\int 0 = C$. (Another true statement, even prettier than the first.) Next, we decree that '+ C' shall be appended to *all* integration rules (just in case). Now, we can all start chanting, 'It is okay to be off by a constant.' But here the Devil steps into the classroom and makes us pay for all that façile cleverness, pointing out...

> As formulated by you, every integral problem now possesses an *infinite number of correct answers*, as we increment C = 1, C = 2... through C = ∞. Therefore, *the* antiderivative of hapless function f is cursed with (supposed) nonexistence, living instead in the limbo of '*an* antiderivative'. You mortals are so intent at creating Hell on Earth, you challenge me to keep up with all your devilish delights!

...or words to that effect. Perhaps Joseph Conrad said it best: 'He argued with himself about all things under heaven with that kind of wrong-headed lucidity which may be observed in some lunatics'. (The passage occurs near the end of his short story, 'An Outpost of Progress'.)

Well, that's the dark view obviously. If you wish to look on the bright side of calculus, it is the field that "wrestles to the ground huge abstract difficulties such as 'infinity' and *tames* them!" (paraphrasing one of my teachers).

There are two reasons for eschewing '∞' in preference to 'eternity':

First, there is the question of linguistic habit: Even if one has read the fine print of the math establishment warning us that '∞ is not a number' (only an unattainable hugeness), there is too strong a tendency to slip into saying '*to* infinity' (as if it *were* a number after all), instead of '*for* infinity', which would be a more suitable phrase, in tune with the aforesaid warranty in fine print. So even if one disagrees with the second point below, because it has overtones of the philosophical (a dirty word in the math and science realm, although both are children of philosophy, thus biting the hand that feeds them), one might consider avoiding ∞ simply on grounds of pragmatism.

Second, there is a more radical point to consider: The kind of process where the mathematician talks about ∞ is not about *quantities* anyway; it is, by its nature, about *time* — eternal time. For tribal and cultural reasons, the mathematics textbook author must remain blind to that fact. But we are free to recognize it and to see where it might lead.

integral

Let's divide this definition into two parts, in chronological order following the student's experience:

1. Brute-force integral. (This is my name for a wide variety of techniques including the Riemann sum, etc.) Definition: *a limit of sums*. Typically, this kind of integral is covered in depth early on in the curriculum. Look at the rectangles in Figure **40** and imagine summing them to approximate the area under the curve. Now imagine using narrower and narrower rectangles until their fit under the curve is nearly perfect. That would be a limit of sums:

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

In short, from this perspective the term 'integral' turns out to be nearly synonymous with *area*. (Notation: The subscripted *i* with an asterisk capping it represents a set of arbitrary sample points for *x*, as in Stewart, pp. 324-325 and Salas, pp. 295-296. In Hughes-Hallett *et al*., p. 252, the symbol c_i is used for this purpose. For yet another notation, see the entry for 'integral sign' on page **196** above.)

2. FTC integral. Later in the curriculum and in most allusions to integral calculus in scientific texts, as signaled by one or more integral signs ∫, ∬, or ∭, the implicit topic of discussion could be the brute-force integral but more likely it is the *FTC integral* — my name for the case where you exploit the FTC (or one of its variants as covered in **Chapter VII**) to discover the limit of sums in a very indirect way (e.g., subtraction of one scalar from another), *not* by actually summing anything: $\int f(x)\ dx = F(b) - F(a)$. Most of integral calculus is concerned with finding more such tricks to evaluate an integral indirectly, to avoid at all costs a tedious brute-force summation.

Comment: When you look at $\int f(x)\ dx$, there seems to be 'a lot going on' there, but for both definition 1 and definition 2 above that left side of the formula is never touched. In the vernacular, you are 'doing an integral' or 'integrating' but all your work concerns, rather, the right side of the formula, not the integrand, $f(x)\ dx$, itself. (Compare the entry for 'Let's integrate'. Note also that the term 'integral' is sometimes used loosely to mean 'integrand', which is listed separately below. See also the entry for the integral sign ∫ in **Appendix F: Symbol List with Annotations**, page **196**.)

integral calculus

The subject taught in Calculus II. In this book, it receives limited coverage, found mainly in **Chapter V**.

integral sign

See page **196**.

integrand

The term 'integrand' denotes the expression inside the intregral sign to be integrated. (Or, occasionally it might denote the expression inside the intregral sign *as* it is being integrated; see Hughes-Hallett p. 352.)

integrate, integration

Please refer to **Chapter V: Integral Calculus**. Synonym: the antiderivative method. See also the entry below for "Let's integrate..." For historical background information, see the reference to Cavalieri on page **132**.

interval

Integration is done over an *interval* of values on the x-axis, not a 'range'. In the vernacular, it might seem natural to use the word 'range' in this context, but in calculus that won't do because 'range' has long since been accounted for with an entirely different meaning that pertains to the y-axis instead! See the definition(s) of 'function' on page **21**.

"Let's integrate f" meaning "Let's evaluate F"

One speaks of 'integrating f' but typically it is the antiderivative F that you work with to actually perform the operation, as illustrated by Figure **20** on page **40**. This same linguistic quirk of pointing at A while thinking about B keeps cropping up in variant forms all through Calculus I, II and III. (In Chinese there is an amusing expression, somewhat related to this practice: 'To point at the chicken while scolding the dog'.)

"Let's differentiate [stuff]" meaning "Let's flag [stuff] for differentiation *someday*"

An author writes, 'Let's differentiate both sides' and follows up with something to this effect: '`d/dx [stuff] = d/dx [other stuff]`'. But for all the noise, so far not one centillion of differentiation has occurred. In fact, the symbol '`d/dx`' is useful for just the opposite of the role implied by 'Let's differentiate [now]'. The symbol '`d/dx`' is invaluable for flagging one's *intent* to differentiate something in the *future*. For an example of this usage, see the discussion of the **Chain Rule: Single Variable** on page **83**. When authors write 'Let's differentiate [stuff]' what they mean is 'Let's *flag* [stuff] for eventual differentiation...'

"Let's evaluate [right side of theorem]" meaning "Let's evaluate left side of theorem"

See entry above for **evaluate**.

LIATE

See page **97** in **Chapter VI: Rules**.

limit

Traditional coverage of 'limit' is provided in **Chapter II**. For an extended discussion of limits, approached from a new perspective, see **Appendix D: Imposed Limits, Inherent Limits**.

line integral

Line integrals are discussed indirectly under the long entry below for 'parametric equations'. (A better name for them would be 'curve integrals' remarks Stewart in passing, on p. 1081. Good point.)

matrix determinant

For an example, see page **145**.

nabla

See the entry for 'del'.

natural logarithm

There is no such thing. What the expression 'natural logarithm' refers to is garden variety logarithms *of* a special number, the number, e. It is the number e that is deemed 'natural', not the logarithms. (A logarithm is a logarithm is a logarithm, which in turn is a fancy name for an *exponent.* As you already know, an exponent is not 'natural' or 'unnatural' or 'fuchsia colored'; it is simply an exponent. Consequently, a logarithm cannot be 'natural' or 'unnatural' or 'kind of mauve'; it can only ever be simply a logarithm.) For more about this, see **Appendix E**.

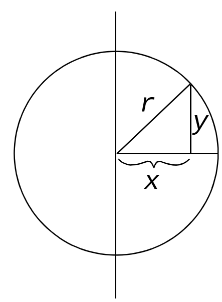
parameterize

In the calculus context, 'to parameterize' means to express a curve in terms of parametric equations, q.v.

parametric equations

Parametric equations come in pairs or trios. This is because they are meant to break down a relatively difficult Line Integral path into two or three constituent functions that are easier to handle. What makes the original path or curve for integration 'difficult'? It could be that the path is has directionality, i.e., that it is made up of vectors. It could be that the path is not the graph of a function. Or, the path over which one wishes to integrate could be both: (a) directional and (b) not a function.

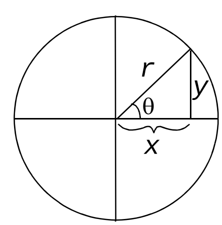
The classic case for illustrating 'not a function' is the equation for a circle, $x^2 + y^2 = r^2$. This may be solved for y, but $y = \sqrt{r^2 - x^2}$ is still not a function since all circles violate the 'vertical line test'. As an end run around this situation, imagine decreeing by fiat that the curve in question shall be defined by a pair of functions that operate simultaneously, one handling the y-values, the other handling the x-values, as follows:



$y = \cos\theta$ OR $y = \cos t$ OR $y(t) = \cos t$

$x = \sin\theta$ OR $x = \sin t$ OR $x(t) = \sin t$

(Assumption: Ours is a unit circle with $r = 1$, so the radius is a nonevent, to be dropped out of the picture.)



Note carefully the alternate notations I've listed above. Implicitly, parameterization is almost always about creating functions with respect to variable t. (In Salas & Hille, the variable u is used for this purpose). Thus, only the third variation in each set above actually makes sense, but the one usually seen in textbooks is the middle form, like $y = \cos t$. By the same token, the expression

'$x = x$' is a used as shorthand for '$x(t) = $ x'. (In words: whatever 'x' was in the original scheme, our new function $x(t)$ will now take on that value.) And so on. For a rare example in which all the implied steps are made explicit, see $e^{y(u)}$ etc. in Salas & Hille, p. 1020.

Now, the only reason we wanted a function in the first place was to take its derivative, as $\Delta y/\Delta x$ or dy/dx or $(d/dx)\ y$ or y' or f' (that's a quick review of some near synonyms in the symbology). So, what is to stop us now from finding dy/dx for the circle? Nothing. We take each derivative separately ($x' = -\sin\theta$, $y' = \cos\theta$, per Table 4 on page 81) then stack one on top of the other, and *voilà*, we're back in familiar territory: $dy/dx = \cos\theta/-\sin\theta = -\cot\theta$.

Once we have obtained f' this way, the path to f'' and beyond is the usual one, no longer involving parametric equations, which were used only to get the ball rolling. Note that one could have taken an entirely different path to the same result, the path called implicit differentiation:

$$x^2 + y^2 = r^2$$
$$2x + 2y\,(dy/dx) = 0$$
$$dy/dx = -2x/2y = -x/y = -\cot$$

(By contrast with all the above, note also that the equation for the *area* of a circle is already a function: $y = \pi r^2$. To take *its* derivative, simply apply the Power Rule: $y' = 2\pi r$, which is to say πD, its circumference.)

For an example that has some context, see note 18 on page 235, where I show the calculations behind the curves in column F of Figure 32 (page 55). The cycloid at the top of column F has a cusp, which breaks one of the rules for differentiability given on page 37. But once the cycloid curve has been expressed as a pair of parametric equations, one can find f' by the 'stacking' method described above (and thence f'' and any higher derivative by the usual methods, all built on f'). Thus, parametric equations bring into the fold a whole new constellation of curves that would otherwise be excluded from the realm of calculus. Likewise polar equations, q.v.

But so far we've barely scratched the surface of all the ways that 'parameterization' is exploited in calculus.

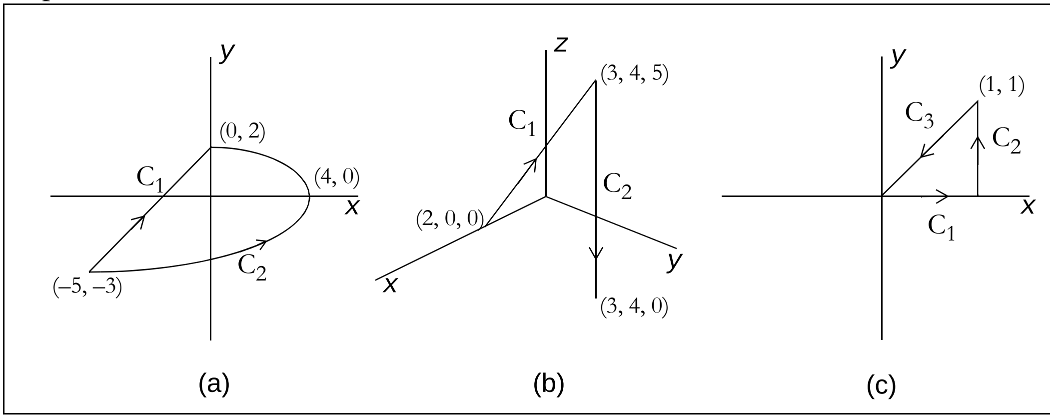


FIGURE 83: Parameterization Examples

Even before performing the breakdown into a separate functions for $x(t)$ and $y(t)$, it is often necessary to first break the curve itself into pieces labeled C_1, C_2, C_3... as illustrated in Figure **83**. As with the circle discussed earlier, the idea is to express movement along the curve by the combined effect of a simultaneous pair of functions. For example, the directed line segment C_1 in Figure **83**(a) may be expressed as $x = –5 + 5t$, $y = –3 + 5t$, $0 \le t \le 1$, where the constants –5 and –3 have been extracted, so to speak, from the original xy coordinates (–5, –3) and where the coefficients 5 and 5 for t are calculated using the final-minus-initial principle: 0 – (–5) = 5, 2 – (–3) = 5. (By convention, the variable t is confined to the interval [0, 1] in parameterizations.) Meanwhile, the parabolic segment C_2 may be expressed as $x = 4 – y^2$, $y = y$, $–3 \le y \le 2$. Segment C_1 in Figure **83**(b) may be expressed as $x = 2 + 1t$, $y = 0 + 4t$, $z = 0 + 5t$, $0 \le t \le 1$, where the constants 2 and 0 and 0 come from the original xyz coordinates (2,0,0) and where the coefficients 1 and 4 and 5 are calculated using, again, the final-minus-initial principle. The three directed line segments of the triangle in Figure **83**(c) may be parameterized as follows:

C_1: $x = t$, $y = 0$, C_2: $x = 1$, $y = t$, C_3: $x = 1 – t$, $y = 1 – t$, $0 \le t \le 1$.

(All the above examples are meant to be fragmentary, to illustrate different flavors of parameterization only, not whole problems. Not shown are the functions for integration, and the step where, later, the two or three corresponding integrals would be summed arithmetically to obtain the desired answer to the multistep problem. For a complete example of parametric equations in context, see the Green's Theorem example discussed on page **117**. For another, see **Verify Stokes** on page 143.)

On the use of the term ‘t-space’: Since translation to parametric equations is done typically with a variable ‘t’, some writers use the term ‘t-space’ as a rough synonym for ‘equations translated to parametric form’, e.g., Bressoud, p. 82-83. To save space, I adopt this practice in Table 7. Related topic: ‘elimination of the parameter’, in Stewart, p. 676.)

Of line equations: the Good, the Bad and the Preposterous:

Let’s take another look at the first half of the parameterization for the line segment C_1 in Figure 83(a). Here is the *specific* parameterization: $x = -5 + 5t$. Recalling where the positive 5 came from and doing a left-right flip on the final expression, we may *generalize* that parameterization as follows:

$$x(t) = \text{ANCHOR} + t\,(\text{FINAL} - \text{INITIAL})$$

(An aside: To get the ball rolling, for the specific parameterization I use the popular style of notation where the instructor writes ‘x’ and hopes/assumes/doesn’t much care if the student realizes that it really means ‘the function $x(t)$’. In the generalized statement, where I am making up my own notation on an ad hoc basis, I switch to the form I prefer where ‘(t)’ is made explicit and with the old ‘x’ replaced by ‘ANCHOR’ to prevent parameter overloading.)

Elsewhere, the student will have learned the following two versions of the vector equation of a line:

$$\mathbf{r} = \mathbf{r}_0 + t\,\mathbf{v}$$
$$\mathbf{r} = \mathbf{r}_0 + t\,(\mathbf{r}_1 - \mathbf{r}_0)$$

When expressed the second way, with the vector relation $\mathbf{r}_1 - \mathbf{r}_0$ spelled out, the equation of a line runs parallel to the parameterization $x(t)$ shown earlier. This relationship clearly ‘means something’ and helps the student understand how all these different ways of a looking at a (straight-line) curve fit together. HOWEVER, the vector equation of a line is often (usually? normally?) shown in the following form instead, arrived at by some algebraic smoke and mirrors:

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\,\mathbf{r}_1$$

From a legalistic viewpoint, one can certainly ‘prove in court’ that the two equations are the same, but in my opinion the popular $\mathbf{r}(t)$ version *means* almost nothing (except to those who can, at a glance, connect all the dots and see that *both* formulas actually mean this: $\mathbf{r}_0 + t\,\mathbf{r}_1 - t\,\mathbf{r}_0$). Meanwhile, the rest of us are left wondering, Why is the $\mathbf{r}(t)$ version pushed so hard, leaving the other version in obscurity? The answer appears to be rather ridiculous: Because that way, you only have to write the $\mathbf{r}$-naught once instead of twice. Gee, how neat is that? (So, it spells convenience for an author or teacher, yes, but for the student, is this true convenience or more of a travesty as the equation of a line becomes just a mindless plug-and-chug exercise? Rough analogy: Suppose my young nephew asks me for a tip calculation formula and I cynically respond with the following: "On your calculator, take the natural logarithm of (Price-Of-Meal divided by 0.8695), then make that result a power of e, *and*, don't forget to subtract out Price-Of-Meal when you're done. There! That's your 15% tip." My hypothetical nephew then becomes addicted to the formula, noting that 'it's easy once you've done it a few times' and 'it always seems to work' but perhaps never divining until years later that it's really just a cruel, nerdy joke. Such is the nature of the ubiquitous $\mathbf{r}(t)$ formula shown above. Always use the $\mathbf{r}_1 - \mathbf{r}_0$ verson instead, the one that mirrors the vector itself.)

partial derivative

See page **90** in **Chapter VI: Rules**.

partial integration

See page **71** in **Chapter V: Integral Calculus**.

polar equation

Like parametric equations (*q.v.*), polar equations open doors on new realms where calculus would otherwise be locked out because of certain rule violations. Starting with a polar equation, one could, for example, take the Second Derivative of the Nephroid of Freeth, which might not prove much, but it would sure sound impressive, and it looks rather interesting, too, with each successive derivative nestled *inside* the closed curve of the previous step, in a descending spiral effect. This gives you quite a different impression of 'what a derivative is'.

power rule

Most of our coverage of the Power Rule is given near the beginning of **Chapter 43**. For a slightly different angle, Table **10** contains some notes about related 'syntax' issues that arise when one tries to talk about calculus instead of simply writing down equations.

TABLE 10: Power Rule, Forward and Backward

POWER RULE	Your Pragmatic Mnemonic	The Rule's Formal Statement	Generic Syntax (in metalanguage)
Forward (for differentiation in Calculus I)	Multiply by the exponent, then reduce the exponent by one.	$(x^n)' = nx^{n-1}$	$P' = Q$ means the derivative of P is Q
Backward (for integration in Calculus II)	Increase the exponent by one, then divide by the increased exponent.	$\int x^n\,dx = \frac{x^{n+1}}{(n+1)} + C$	$\int Q = P$ means the antiderivative of Q is P (i.e., P is simply the *function* we began with).
English language P/Q trap: Note how the following statements, natural though they are in *speech*, run against the left-to-right grain of the two *formulas*' syntax: "Q is the derivative of P" and "P is the antiderivative of Q". To avoid confusing yourself, always recite these rules left-to-right, as shown in column 4. See also the entry for $\int$ Q = P on page 196.			

rise over run

See Figures 7 and 8 in **Chapter I: Slopes and Functions**.

slope

See Figures 7 and 8 in **Chapter I: Slopes and Functions**.

t-space

See the entry for 'parametric equation'.

u-substitution

See **Integration By Substitution (alias 'u-substitution' or 'w-substitution')** on page 92.

vector calculus

In vector calculus, integrals are taken over objects that possess not only magnitudes in 2D or 3D Euclidean space, but also a direction, such as 'clockwise'. In many curriculums, the term 'vector calculus' is synonymous with 'Calculus III'; sometimes it is used also as a synonym for 'multivariable calculus'. In this volume, a long chapter (**Chapter VII**) is devoted to vector calculus topics. By contrast, in most calculus ‘companion books’, this huge topic is treated as if it does not even exist.

w-substitution

See **Integration By Substitution (alias ‘u-substitution’ or ‘w-substitution’)** on page **92**.

wonk

A hardworking expert who fixates on the minutiae of an issue or problem. E.g., a ***policy wonk*** in D.C. is the back-room person, possibly a nerd or geek, who will work out all the technical details of implementing a newly proposed political scheme. I propose the term ***wonk calculus*** as a half-serious replacement for ‘modern calculus’. I object to the term ‘modern’ on principle since something modern (such as modern music) stays modern only until it is passé (or until history reaches its graceful conclusion and all the clocks stop ticking). By contrast, the wonk, for better or worse, is here to stay. And yes, I am slightly suspicious of ‘modern calculus’, somewhat underwhelmed by the whole thing, and that is another reason I eschew the term. (For more about this, see notes **2** and **3** on page **229**.)

Linguistic sidebar on the closely related word ‘geek’: The Chinese word for ‘person’ is 人 *ren*, which remains ambiguous as to gender. So, in that culture one may (generally) avoid any chest-thumping and teeth-gnashing about gender-biased language (*mankind* versus *personkind* and all that). Just like a Chinese *ren*, so a Western geek seems ambiguous as to gender, suggesting a computer or math expert who is either male or female. So it seems that we have finally caught up to the Chinese, at least where that one word is concerned (although a ‘wonk’ strikes me as a distinctly male type, not gender-ambiguous). However, somewhere in the second or third decade of the twentieth century, the Chinese decided to be ‘modern’ and added a three-way distinction in their written language: Today, there is still the one sound, *ta*, but one has a choice (strictly optional) of modulating its written form to reflect gender, using 他 (he), 她 (she), or 它 (it). (By contrast, there was previously a single character 他 covering all three cases, thus mirroring exactly the spoken language.) So, in the global perspective, it looks like One Big Step Forward for Geekdom, One Little Step Backward for Chinese He’s and She’s and It’s.

Are these terms *disparaging*? Quite possibly. But the author of this book is a self-described wonk (in the field of sinology, Harvard Ph.D. 1975, *not* mathematics, I hasten to add).

Notes

Numbering: The notes are numbered continuously through the prologue, chapters, and appendices. (I.e., there is only one instance of ‘note 1’.) To make them somewhat bidirectional, I precede each note by an italicized excerpt from the passage it is attached to. (In this practice, I am mimicking the style seen in Mandelbrot and Hudson [2009, 2004].)

1 ‘*The power and elegance of Lagrange’s prime notation...*’ (page **3**)
In fact, one might even say prime notation is *too* elegant and powerful, so that the temptation to ‘overload’ it becomes irresistible: Thus it carries two dramatically different definitions, dependent on context:
-- the derivative at a given *point* in function f (i.e., one specific slope).
-- the derivative *function* for the entire *ensemble* of points that belong to function f; see pages **27**-**29**.

2 ‘*The picture I’ve just painted is essentially the truth*’ (page **4**)
Not that I’ve quoted anyone in particular. With my pseudo-quote, I’m just trying to convey the tone of various reviews I’ve read of Spivak’s *Calculus on Manifolds* and Apostol’s two volumes of *Calculus* and Arnold’s, *Mathematical Methods of Classical Mechanics* and the like. In short, to the math major, there exists a kind of ‘alternative calculus universe’ that we need to acknowledge at some point. The expected, polite term for that alternative universe is ‘modern calculus’. My reasons for calling it ‘wonk calculus’ instead are given on page **227**. (By the way, the term ‘manifolds-with-corners’ is real; I lifted it from the grand finale in Spivak, p. 137.)

3 “*From here on I will use the term ‘calculus’ to allude usually to the vintage calculus milieu with all its special problems, not to the wonk calculus milieu*” (page **4**)
Exception: I will however make several references to Bressoud’s book. Why Bressoud? While taking Calculus III (vector calculus), I worked through Bressoud’s book on my own. Its title is *Second Year Calculus*, and on the back cover it says, ‘This is a textbook for multi-variable and vector calculus’. Those are both red herrings, suggesting to the naive customer something very different from what the book actually is. Rather, it is intended as the vehicle that will take you *away* from the world of Calculus I

through IV, in the direction of wonk calculus instead. How did *I* get through the book, then? In my ignorance, when I ran into something like Bressoud's "[exploitation of] the ambiguity between points and vectors" (p. 21) or the FTC expressed as $\int_{\partial M} \omega = \int_M d\omega$ (p. 283), I put it down to a cocktail of eccentricity, willfulness and ego — something like Norbert Wiener's manner of 'writing for the public' — and kept on trudging since there are such slim pickings out there for the [actual] student of Calculus III. Long story short, it turns out that by chance there are a few parts of Bressoud's book that *do* translate reasonably well back in the direction of our humble vintage calculus. In fact, I became so enamored of his two-dimensional fluid flow example (Bressoud, p. 84) that I decided to use an enhanced and prettified version of it as the capstone to my own **Chapter VII**, i.e., for wrapping up the whole book; see page **137** above. (For very different reasons, I've also included a half-dozen minor references to Bressoud, not always so complimentary.)

4 '*80 percent algebra and only 20 percent calculus*' (page **5**)
In this special context, 'algebra' is not taken literally but is understood to mean 'elementary mathematics'. And 'elementary mathematics' (as defined in Salas and Hille, p. 1, for instance) means 'geometry, algebra and trigonometry'. Within the numeric content of the curriculum, it is often difficult to draw the line. E.g., in the difference quotient (introduced on page **27**) one may suspect an instance of '80% algebra and 20% calculus' but in that context the two threads are hopelessly entangled. By contrast, the subtext that I intend for **Chapter IV: Curves!** is: 'These belong wholly to the '20% calculus' component.'

5 '*Berlinski's notion of curves as the "faces" of their respective functions*' (page **10**)
Cf. Chiang Yee, p. 176-177, where he introduces certain Chinese characters 'of which neither part faces either inwards or outwards but both face forwards'. I.e., while many Chinese characters have component parts that seem to be standing on a stage 'in silhouette', some characters are notionally turned 90 degrees toward the audience: these ones have 'faces'! Chiang's examples: 願 and 體 , the characters for 'wish' and 'body'.

6 '*If we look back at precalculus, it too has a split*' (page **11**)
At the next level down, one should distinguish those precalculus books that contain a pre*view* of calculus (e.g., by way of the difference quotient, as in Hungerford, p. 139) from those precalculus books that cover elementary mathematics *only* (such as the book by Leff and the one by Simmons).

7 ‘*Make a pile of six pebbles, for instance*’ (page 17)
The first sentence on page 1 in Salas & Hille: “To a Roman in the days of the empire a ‘calculus’ was a pebble used in counting and in gambling.” By contrast, my use of pebbles in Figure 7 was serendipitous, with no conscious intent to resonate with the etymology. (Subconscious working overtime?)

8 ‘*The conventional version looks more “natural” than the real thing*’ (page 19)
In fact, if you plot the zero-to-π portion by itself, by hand, without its companion curve beneath the x-axis, it has a ‘tortured’ feel about it. Try this using the following *xy*-coordinates and you’ll see what I mean: (0.31, 0.31), (0.63, 0.60), (0.95, 0.81), (1.20, 0.95), (1.57, 1.00), (1.90, 0.95), (2.20, 0.81), (2.50, 0.60), (2.80, 0.31), (3.14, 0.00). These are exact coordinates, down to the one-hundredth of a unit, so it can’t be a case of ‘missing the curve’. Yet the curve they *do* trace out has something decidedly ‘unnatural’ or awkward about it, by comparison to the many other graphs of functions that possess an immediate appeal to the human eye. If it were a person, the sine wave might be a ‘troubled child’.

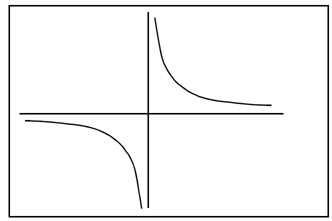
9 ‘*In* Figure 14*, associated now with functional notation*’ (page 25)
Figure 14 is based loosely on Exercise 14 in Hughes-Hallett *et al.*, p. 79, where a generic (nonspecific) function is used. In Figure 14 we’ve rewritten the exercise in terms of a specific function, $f = \ln(x)$, by way of preparing the reader for Figure 20 on page 40 which involves both $f = \ln(x)$ and its derivative function, $f' = 1/x$.

10 “*It is the variable* h *that tells us we are ‘not in Kansas anymore’* ” (page 27)
As detailed in the ensuing paragraphs, the ‘not in Kansas’ aspect is true even for the difference quotient in its brute force usage where a value such as 0.0001 assigned to h, and it becomes especially clear when $\lim_{h\to 0}$ is slapped on the front of the expression and h is held in abeyance until the very last step of the operation for calculating a derivative function, at which point we assign it the value 0.

11 ‘*We’ve shown the case where* x *lies to the right of* c’ (page 35)
A quick survey of the literature as regards the obligatory δ and ε picture of limits: Not surprisingly, Salas & Hille do it with loving care, devoting a whole *sequence* of illustrations to the idea (Salas & Hille, p. 58-59, Figures 2.2.3 through 2.2.5). Stewart, too, tells the whole story, although at first glance one may think that x and y are missing: Rather than drawing lines (such as the dotted lines in Figure 18), he places a terse dot on each axis to represent x and y (Stewart, p. 95, Figure 1). In Hughes-Hallet et al., p. 50, we encounter a stripped down version in which x and y are simply

forgotten. This occurs in Berlinski, too (p. 153), and in quite a few other books, one suspects. The stripped down version is probably a case of "The path to Hell is paved with good intentions." No doubt the author strips it down from fear of burdening the reader with too much detail. But absent x and y, the picture becomes nearly worthless to the *reader* (so then who cares if the *author* finds it prettier?) Speaking of pretty, Messrs. Keppner and Ramsey have contrived somehow to be both incoherent *and* ugly, in Quick Calculus, p. 57. They get the prize for Worst Conceivable Delta-Epsilon Mess.

12 '*To be treated as a function in its own right*' (page 39)
To juxtapose $f = 1/x$ with a single curve in the northeast quadrant is actually a half-truth. $f = 1/x$ is known as 'the reciprocal function', the graph of which has these two parts:



The part that occupies the southwest quadrant is typically discounted when one's context is the $y = \ln(x)$ function.

13 '*By a simple measurement of its height with a string*' (page 46)
Possible point of confusion: In contrast to a common type of trig problem where a shadow is easily measured but the object casting it is too tall to measure directly, we are imagining, by fiat, a situation where measuring the tree would be relatively easy but measuring the area of its shadow directly would be tedious. Just the opposite.
Switching gears entirely now, here is something extra that is related to Figure 24 though not relevant to the discussion in the text: In case we are curious to see what the whole $F = \ln(x)$ curve looks like after the 90 degree rotation, here it is drawn against axes labeled L (for logarithmic

or linear) and M. It is thus reincarnated as the 'straight-line curve', $L = \ln(M)$.

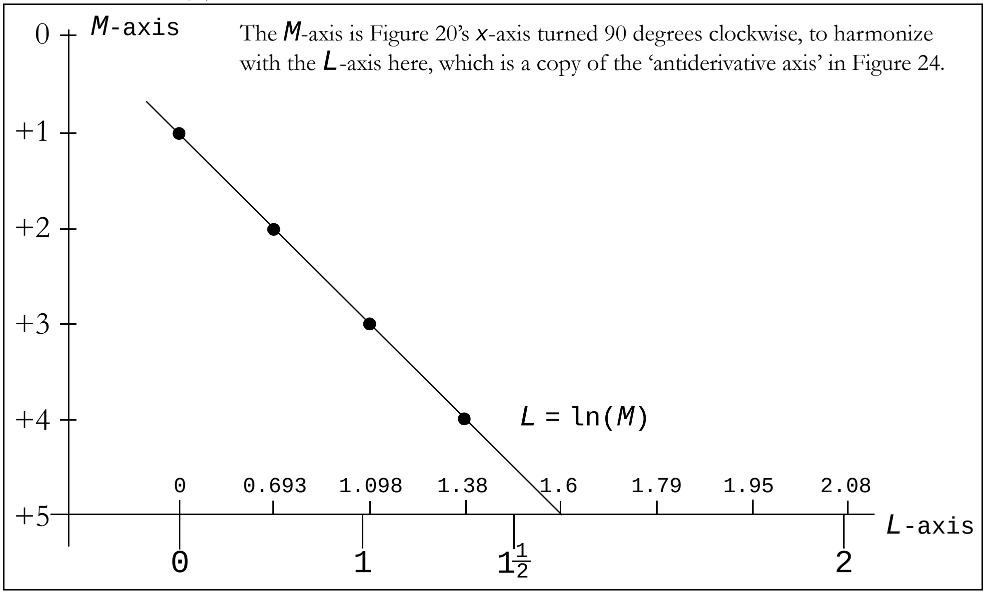


FIGURE 84: Logarithmic Function Turned 90 Degrees

14 '*The Curvature Kartouche, which is depicted in Figure 25*' (page 48) Technical notes about the three curves in Figure 25: The top curve represents the function $y = 1/(1 + x^2)$. Here, we choose to use that function as f, our *primary* function. (Its curve happens to be also that of the first derivative of $y = \arctan(x)$, as represented in **Table 5, "The Trig Inverse Rules," on page 82**, for example, but in this context we do not exploit that relationship.) The two lower curves are the first and second derivatives of $y = 1/(1 + x^2)$. At the far left I've taken some poetic license: In that region, I've forced the f'' curve down slightly so that it falls below the f and f' curves, for the sake of a pattern that is more coherent and pleasing to the eye. (Strictly speaking, the left tail of the f'' curve should be squeezed in-between the left tails of the f and f' curves. The actual shapes can be seen on a TI-83 calculator using the TRACE feature with the Zoom Decimal option, and with the Window set to –3, 3, –2, 1.5.)

15 '*This chapter is devoted to the enjoyment of their beauty*' (page 53) As mentioned on page 6, in its early stages the working title of this book was *Calculus as the Language of Curves*. Over time, an infusion of new materials and themes into the manuscript meant that the original title would no longer fit. (Ultimately, the grandiose-sounding notion of a 'language of curves' was reduced to the confines of **Chapter IV**, with some help from Chapters **III** and **V**.) Beyond that primary motivator for a

title change, there is a certain irony to note as well: Calculus is the subject which — in a sense — denies the existence of *any* curve, anywhere, by breaking it down always into a multitude of tiny tangential line segments. Perhaps that too discouraged me from keeping the original title, at least on a subconscious level. But on the conscious level the inherent contradiction didn't dawn on me until rather late in the project, long after I had changed the title for the other reason mentioned above.

16 '*Each trio (labeled A, B, C...) is a set of intimately related curves*' (page **53**)
The drawings in Figures **31** and **32** are my own, but they are based largely on graphs that I excerpted and rearranged from Robinson *et al.*, pages 69-70 and 78. The cycloid group (column F) I put together based on hints found at planetmath.org.

17 '*The number of curves in such a family is unlimited*' (page **55**)
In fact, if you study the relation between trio C and trio D, you'll see that curves 2 and 3 of trio C reappear as curves 1 and 2 of trio D, suggesting that perhaps curve 3 of trio D is simply the fourth member of trio C's family, which it is: that curve could be appended to column C and labeled f'''. Along the same lines, note how the final two curves in the table match the first two curves. This resemblance is not random.

18 ‘*That’s the nonintuitive curve shown at the top of column F*’ (page 56)

The cycloid itself is modeled using a pair of parametric equations, as follows:

$$x = a(\phi - \sin\phi),\ y = a(1 - \cos\phi)$$

Even though we don’t have a function to differentiate in the normal way, we have the ingredients required to define the cycloid’s first derivative as dy over dx, which gets us to the same place:

$$\frac{dy}{dx} = \frac{\frac{d}{d\phi}(1 - \cos\phi)}{\frac{d}{d\phi}(\phi - \sin\phi)} = \frac{\sin\phi}{1 - \cos\phi} = \cot\left(\frac{\phi}{2}\right)$$

Thus far, I have followed a cycloid article at planetmath.org/encyclopedia, accessed 08/26/10. Some details of the first derivative calculation follow:

We assume a will be set to 1, and ignore it for the duration.

By Constant Rule and Trig Rules, the derivative of $(1 - \cos\phi)$ is $0 - (-\sin\phi) = \sin\phi$.

By Power Rule and Trig Rules, the derivative of $(\phi - \sin\phi)$ is $1 - (\cos\phi)$.

The final step relies on the following Half-angle Identity:

$$\tan\left(\frac{x}{2}\right) = \frac{1 - \cos x}{\sin x}$$

By flipping that trig identity over, we obtain the identity shown for $\cot\left(\frac{x}{2}\right)$

Second Derivative: Take the first derivative of $\cot\left(\frac{x}{2}\right)$.

Begin by noting that $(\cot x)' = -\csc^2 x$ where $-\csc^2 x$ means $-\left(\frac{1}{\sin x}\right)^2$

Thus, by chaining of the Trig Rules,

$$\left(\cot\left(\frac{x}{2}\right)\right)' = \left(-\csc^2\left(\frac{x}{2}\right)\right)\left(\frac{x}{2}\right)' = \left(-\csc^2\left(\frac{x}{2}\right)\right)\left(\frac{1}{2}\right) = \boxed{-\frac{1}{2}\csc^2\left(\frac{x}{2}\right)}$$

In practical terms, our answer translates to the following on a TI-83 calculator:

Y1 = (–) 0.5 (`1/sin(x/2))^2`

With the window set to Xmin = –2π, Xmax = 2π, Ymin = –20, Ymax = 1, one should see a pair of inverted capital U shapes. That’s the second derivative. The first derivative was the more familiar cotangent curve(s).

Backing up to the cycloid itself, that can be displayed by setting MODE to PAR (parametric) and entering 1.0(T – sin T) and 1.0(1 – cos T), with window set to Tmin –3π, Tmax +3π, Tstep 0.5, Xmin –10, Xmax 10, Ymin –1, and Ymax 3.

19 ‘*In Figure 38 I’ve sketched my impression of the two images he evoked.*’ (page 63)
I would be hard-pressed to say why, but my instructor’s speedometer/odometer example lingers in memory, as though it were something poetic. Looking back on my two years of calculus, many exotic concepts and images come to mind: Gabriel’s Horn (explored at some length in Boyce, 2010a, pp. 237-263); the Nephroid of Freeth (sounding like a companion piece to H.P. Lovecraft’s *Fungi of Yuggoth* but actually, a special kind of limaçon, as presented in Stewart, pp. 701-704); and so on. But there was something special about that moment of the broken speedometer / broken odometer in Gerry Naughton’s class, in the Fall of 2005. (In a similar vein, Priestley uses a rocket ship speedometer example [pp. 216-217] to introduce the antiderivative method (of integration) [p. 234], and a motorcycle speedometer example [pp. 234-236] to help show the nonintuitive “connection between the calculus and the calculation of area” [p. 234].) Also, I was impressed by the way Dr. Naughton had distilled a subject as large and complex as integral calculus down to a single expression, $\Delta x \bullet \Delta y$. In Chapter I, I pointed out that the hypotenuse of a right triangle, beyond its role in the pythagorean theorem, is also a ‘picture of division’ hence a representation of ‘slope’, even an icon that could stand for ‘differential calculus’. Here we encounter the analogous idea for integral calculus: Look at a rectangle and what do you see? Beyond its conventional role in geometry, it is also a picture of $\Delta x \bullet \Delta y$ in the generic sense. Now imagine a series of rectangles arranged under a curve, and the sum of their specific $\Delta x \bullet \Delta y$ values amounts to a kind of rough-and-ready integration. Viewed in this light, the expression $\Delta x \bullet \Delta y$ is indeed pregnant with meaning, a reasonable stand-in for ‘integral calculus’.

20 ‘*For the most part, the antiderivative is represented by the dark vertical bar running from 21 to 72 along the y-axis*’ (page 68)
The dark vertical bar in Figure 41 is analogous to the dark horizontal bar(s) in Figure 24 on page 45, where I call that axis the ‘Shadow Axis’ after rotating it 90 degrees.

21 ‘*Bounds of integration, 0 to 1 along each axis*’ (page 71)
The example that follows is extrapolated from my notes on a lecture given by Mihail Cocos near the beginning of his vector calculus class at University of Minnesota, 10-4-06.

22 ‘*The presentation proper begins with* Figure 54’ (page 103)
A minor departure from Wood’s presentation: I differentiate two *versions*

of Green's Theorem where Wood speaks of two *forms* of Green's Theorem. I've switched to the term *version* only to avoid a nomenclature clash with *form* in my higher-level distinction between FTC Canonical Form and FTC Form in Figure **53**.

23 *'An example of the circulation version of Green's Theorem follows'* (page **114**)
The example is based on problem #1 in Stewart, p. 1108. My version uses a smaller rectangle and a simpler function. I've also introduced a sign change to force all of the piece-wise integrals to be positive so that both parts of the 'fence' stay above ground. My intent is to show something like the rock bottom minimal case for a 'fence on a line integral', as the idea pertains to Green's Theorem.

24 '*In wonk calculus the letter* d *is employed to guarantee visibility at all times for this pattern that threatens to slip into the background of the thorny notations of vintage calculus'* (page **118**)

$$\int_M d\,\omega = \int_{\partial M} \omega$$

For the aspiring wonk, I've reproduced the magic formula that accomplishes this feat at the right. This formula appears in Spivak, p. 124, as Stokes' Theorem (in the role of 'star of FTC variants' so to speak), and on Bressoud's cover and in Bressoud, on p. 283, as *the* FTC itself (!) Which it is, if you think abstractly, along the lines of Stewart, p. 1134: 'Notice that in each case we have an integral of a "derivative" over a region on the left side, and the right side involves the values of the original function only on the *boundary* of the region' (his italics). This passage (which I quote for a similar reason in note **25** and for a very different reason on page **211**) occurs near the end of Stewart's vintage calculus textbook, as he tries to smuggle in a bit of enlightenment from the wonk calculus viewpoint, we suspect.

25 '*There I present an expanded version of the vector calculus summary table found in Stewart'* (page **119**)
On p. 1152, Stewart presents an excellent summary table, prefaced by a reminder to the student that even the Green, Stokes and Divergence Theorems are all just variations on the FTC theme:

> Notice that in each case we have an integral of a "derivative" over a region on the left side, and the right side involves the values of the original function only on the *boundary* of the region.— Stewart, p. 1152 (emphasis in the original)

His distillation is much appreciated by the student, not only in its intended use as a study guide for the Calculus III final exam, but equally as a preview of the course. *However*, to my way of thinking, his table has an essential column missing, the one where I would want to classify each theorem first in terms of its geometric identity. See the following table where I've reproduced the essentials of Stewart's summary in columns 1 through 3, with the implied geometric identities appended as a fourth column.

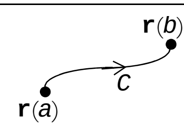
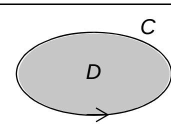
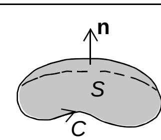
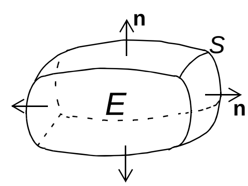
Name(s)	Theorem	Sketch after Stewart p. 1152 (minus his color coding for boundaries)	Implied Geometric Identification
Fundamental Theorem of Calculus	$\int_a^b \mathrm{F}'(x)\,dx = \mathrm{F}(b) - \mathrm{F}(a)$		The *function* resides in a 1D space. (No hint given of the *derivative*'s realm.)
FTC for Line Integrals	$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$		The *function* is scalar. (No hint given of the *derivative*'s realm.)
Green's Theorem	$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy$		The *function* resides on the 1D boundary of a 2D region. (By chance, this provides a hint of where the *derivative* resides, too.)
Stokes' Theorem	$\iint_S \mathrm{curl}\,\mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$		The *function* resides on a space curve. (Incidentally, this provides a hint of where the *derivative* resides.)
Divergence Theorem (Gauss's Theorem)	$\iiint_E \mathrm{div}\,\mathbf{F} \cdot dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$		The *function* resides on a 3D 'skin'. (Incidentally, this provides information about the realm of the *derivative*.)

FIGURE 85: The Five Basic FTC Variants after Stewart, page 1152

Notation: If you are curious about Stewart's unusual `F'/F` combo in row 1 (where f/`F` is expected) and/or the *del* notation in row 2, please refer to page 193. (The latter is probably his rationale for the former.)

From the wording I use in the appended 4th column, it may seem that I am reading a lot into Stewart's table. Here is part of the context: In Bressoud, the FTC is characterized as the one 'for scalar functions of one variable' (Bressoud, p. 279). In Protter and Morrey, Green's Theorem is characterized as follows: 'Green's Theorem is an extension to the plane of the Fundamental theorem of calculus' (Protter and Morrey, page 445). There is a general unstated assumption in both vintage and wonk calculus that the FTC or one of its variants is to be classified, geometrically, solely in terms of the *function* (on the right side, when expressed in FTC Canonical Form), while the part that is 'merely' the *derivative* (on the left side, when expressed in FTC Canonical Form) is ignored.

Meanwhile, I advocate the opposite approach. An $n+1$ dimensional description, focused on the left side, tells you everything you need to know, whereas an n-dimensional description, focused myopically on the right side, can only lead to an unsatisfactory characterization. The result of doing it 'my way' is shown in Figure 63.

Sources of descriptions in Table 7: The description of Line Integral with respect to Arc Length/Space Curve is mine; it needs to be read in conjunction with Figure 63. The description of Stokes' Theorem I've cobbled together from Stewart, p. 1139 (boundary as space curve) and Schey, p. 93-94 (capping surface). The description of Gauss's Theorem is my own: picture a solid, E, that possesses a surface, S, which is a kind of skin that is 'aware' of its 3D nature, and whose data necessarily spill over into a fourth dimension.

26 '*At first glance it may seem that I am attempting something distantly related to manifolds*' (page 121)
My '1D-in-2D' tag and '2D-in-3D' tag may appear to be related somehow to the objects shown in Spivak's Figure 5-1, where the caption reads: "A one-dimensional manifold in $\mathbf{R}^2$ and a two-dimensional manifold in $\mathbf{R}^3$" (Spivak, p. 110). But my own scheme (worked out when I was still in complete ignorance of manifolds) is biased toward classifying objects *in* certain spaces while the manifold language seems biased toward turning objects *into* spaces, *and* vice versa! Below are two excerpts from the wikipedia entry for 'manifold' (accessed 09/27/10).

> A mathematical space that on a small enough scale resembles the Euclidean space of a specific dimension, called the dimension of the manifold. Thus a line and a circle are one-dimensional manifolds, a plane and sphere (the surface of a ball) are two-dimensional manifolds, and so forth...
> Carl Friedrich Gauss may have been the first to consider abstract spaces as mathematical objects in their own right. His theorema egregium gives a method for computing the curvature of a surface ***without considering the ambient space in which the surface lies.*** Such a surface would, in modern terminology, be called a manifold; and in modern terms, the theorem proved that the curvature of the surface is an intrinsic property. Manifold theory has come to focus exclusively on these intrinsic properties (or invariants), while largely ***ignoring the extrinsic properties of the ambient space.***

The passages quoted above (my italics) should help explain why I say the manifold concept is roughly the 'opposite' of what is needed for my narrow purpose.

27 '***Here is a 1D object embedded*** in ***a 2D region, here is a 2D surface embedded*** in ***a 3D space***' (page 121)
The following table contains some analogies for the various j-dimension-in-k-dimension types:

Pure 0D	A bead
0D-in-1D	A bead on a fine steel wire
Pure 1D	A fine steel wire
1D-in-2D	In your mind, there is a straight, 3-inch piece of string somewhere in the room, but at the moment it is coiled up, occupying a 2D region on a shelf.
Pure 2D	A flat piece of paper
2D-in-3D	The ground you stand on, seemingly flat, actually a tiny patch on the 3D globe.
Pure 3D	A jet fighter pilot, with 'situational awareness' of her x, y and z coordinates at a given moment.
3D-in-4D	The same pilot, moving through time. Note that some modeling requires the notion of '*a* fourth dimension', not always '*the* fourth dimension'. (See page 135.)
Pure 4D	A Mozart symphony, said to have been viewed by Mozart himself as a static, 4D solid wherein he could travel to and fro at will.

28 ‘*Thus seeded with zeros, the third dimension has sprouted a nine*’

$$\langle abc\rangle \times \langle def\rangle = \begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix} = (bf - ce,\ cd - af,\ ae - bd)$$

(page **138**) The cross product is computed using the rule shown above (after Bressoud, p. 147), as follows: $\langle 4, 1, 0\rangle \times \langle -1, 2, 0\rangle$ $=[(4 \bullet 0)-(0 \bullet 2), 0 \bullet (-1)-(4 \bullet 0), (4 \bullet 2)-(1 \bullet (-1))] = (0, 0, 8+1) = (0, 0, 9)$.

29 ‘*Thus, one of the major epiphanies of elementary calculus is missed*’ (page **149**)
But *does* the appearance of ‘$4\pi r^2$’ in Figure **46** deserve so much attention? I confess that the V-to-V' relation for the volume and surface of a spherical ball no longer gives me the thrill it once did (when I first encountered it, about five years ago). Currently my reaction to it tends to be more along the lines of: “It’s just a higher dimensional version of $A = \pi r^2$ and $A' = 2\pi r$, where the derivative of a circle’s area turns out to be the circle’s own circumference. Given that we believe in the Power Rule, what else could the numbers do?” And this in turn makes me think of Euler’s formula, $e^{i\pi} = -1$, “by many regarded as the most beautiful formula in all of mathematics” (Gullberg, p. 507). Or could it be that Euler’s formula is only an inevitable tautology, where several of us earthlings’ favorite symbols find a way to circle back upon themselves in uroboros fashion (a tail-biting exercise)? That’s the alternative view that nags at me. Long ago I went through my own phase of possessing a mouse’s awestruck admiration for the monumental equation, but nowadays my feeling toward it might best be described as agnostic.

30 *‘the inherent deficiency of a macroscopic being in dealing with time or space on the cosmic scale’* (page **172**)
One of the few who bothers with trying to bridge the gap is Emily Dickinson, as reflected in the following excerpts:

Next time, to tarry,
While the Ages steal —
Slow tramp the Centuries,
And the Cycles wheel! [from ED 160]
Since then — tis Centuries — and yet
Feels shorter than the Day
I first surmised the Horses Heads
Were toward Eternity — [from ED 712]

Yes, ED has more on her mind than just swooning over the passage of Centuries: “Voyeurism, vampirism, necrophilia, lesbianism, sadomasochism, sexual surrealism: Amherst's Madame de Sade still waits for her readers to know her." (Sources: Richard Ellmann, ed. *The New Oxford Book of American Verse*, pp. 325 and 342-343. Camille Paglia, ***Sexual Personae: Art and Decadence from Nefertiti to Emily Dickinson*** [Vintage, 1990], p. 673.) Similarly, the improper integral still waits for *her* readers to know her, as she inches along *for* eternity, not *to* infinity!

31 '*The curvature is in fact alive and well, continuing for eternity*' (page 175)
How does one find a needle in the haystack such as our 'window' residing in the minuscule space between $y = 0$ and $y = 10^{-13}$? No doubt there are numeric methods that could be exploited for the purpose, but for populating the eight windows near the bottom of Figure 73 I resorted to trial-and-error, guided at times by a modest amount of 'number sense' to find the y-axis values I needed to make the line reappear on my TI-83 display. Then I chose a pair of x-axis values that would bring out the never-ending curvature of the line.

32 '*This problem statement is after Hughes-Hallett...*' (page 175)
It seems safe to assume that Hughes-Hallett *et al.* picked up this function from an actual study of soot distribution, rather than invent it out of thin air. Hence my rather tenuous claim to have switched here to a 'real' example.

Units: In their problem statement, they introduce the function using an unseemly conflation of kilometers and millimeters:
"The depth, $H(r)$, in millimeters, of the soot deposited each month at a distance r kilometers from the incinerator is given by
$H(r) = 0.115e^{-2r}$" (Hughes-Hallett *et al.*, p. 398, #11).
I prefer to state the density function in terms of a single unit, kilometers, which can be accomplished easily enough by inserting some zeros:
$H(r) = 0.000000115e^{-2r}$. Granted, *eventually* it makes sense to show *km* in the x-direction and *mm* in the y-direction (as I've done in Figure 74a, for example). But at the outset the mixed units rub me the wrong way.

33 TI-83 calculator notes for Figure 74b (page 176):
'Y=' 1.15E–7(e^(–2x))2πx
where the 'Y=' key defines the function, and we let r be represented by x.
Window: 4, 5, 0, 1.5 x 10^{-9}

34 TI-83 calculator notes for Figure 75 (page 177):
'Y=' 1.15E–7(e^(–2x))2πx
where the 'Y=' key defines the function, and let r be represented by x.

Same equation as for Figure 74b, but with different window values:
Window: 0, 5, 0, 10^{-6}
TRACE (with *x* chosen visually to hit close to the highest *y* value):
When $x = 0.5$, $y = 1.33 \times 10^{-7}$ *km*, say 0.1 *mm*

Double check on manual evaluation of the integral:
2nd CALC, $\int f(x)dx$, Lower Limit $x = 0$, Upper Limit $x = 5$
$\int f(x)dx = 1.8055\text{E}{-7}$
This matches our 1.805×10^{-7} km^3 which we converted to 181 m^3.

35 ‘*Now with the two dimensions cranked up a notch*’ (page 177)
Note also the parallel with the ‘sphere epiphany’ (page 148): The derivative of the V_{CYLINDER} function is none other than the equation for the surface of a cylinder. But unlike a uniformly curved ball, this particular object composed of soot is shifting to something ever slightly wider and ever slightly shorter (*or* taller). This makes it more difficult to construct a model that will illustrate the V-to-V' relation for the constituent cylinders. But underneath it all, the same kind of derivative relation holds as between a ball and its spherical shell, or between a circle and its circumference. *If* the soot formed a single ‘bump’ at the center, instead of rising to a ‘wall’ at the 1/2 *km* mark, then the integration of its volume could be modeled with a set of myriad concentric cylindrical shells, most of them too short to have a profile on the horizon. In Figure 86 I’ve sketched out a crude suggestion of how the latter type of model *would* work in a simpler scenario than ours, e.g., tornado siren audibility.

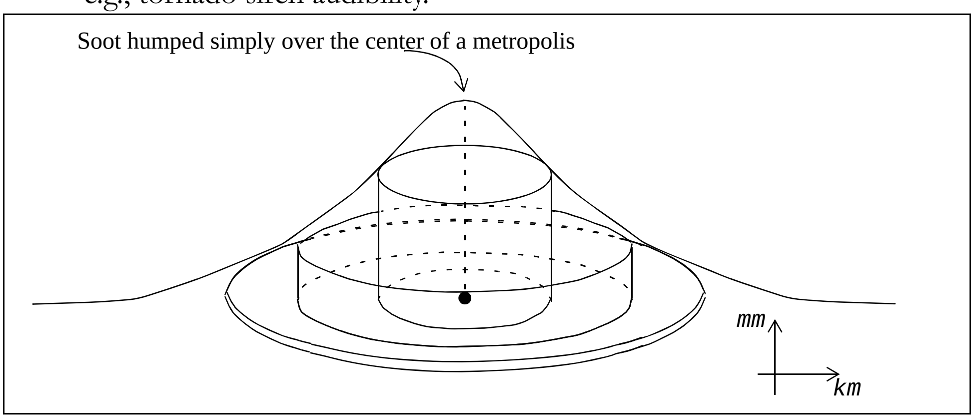


FIGURE 86: A Simpler Soot Scenario

36 ‘*To express them in simple rustic numerals is more becoming of an ignorant biped earth-dweller*’ (page 200)
For that matter, even ‘1/3’ as ‘exact’ and ‘0.333’ as ‘approximate’ is presumptuous, in the same way π and *e* are presumptuous. From the biped

viewpoint, one third is a very mysterious concept, one that literally has no definition. (One fifth, by contrast, is tangible, as two tenths.) I remember ‘like yesterday’ the moment in grade school, probably circa 1950, when I first tried drawing a circle divided in thirds. On the one hand, ‘it worked!’ — that trick of simply writing the letter ‘Y’ inside a circle then coloring the perfect thirds. On the other hand, there was something slightly spooky about it, as though one had communicated with an alien intelligence or stepped outside the bounds of decency for a mere biped. Some 60 years on, I can recall that event but much of its magic is gone. The lost magic has precisely to do with treading in the realm of the *exact* where a glorified simian has no business putting on airs. In that sense, a seven-year old has insights into the nature of ‘exact’ and ‘approximate’ that are sometimes more attuned to reality than those of the adult mathematician (or adult math dilettante in my case).

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Nykamp, Duane. "The idea behind Green's Theorem" 4 pp. http://www.math.umn.edu/~nykamp/m2374/readings/greensidea/index.html accessed on 07/19/06.

Priestley, William McGowen. *Calculus: A Liberal Art* (Springer, Second Edition, 1974 [1998]) 401 pp.

Protter, Murray H. and Charles B. Morrey, Jr. *A First Course in Real Analysis* (Springer-Verlag, Second Edition 1991 [1977]) xviii + 534 pp.

Robinson, Maria, Christopher. Goff, Beverly K. Michael, and Peg Kem McPartland. *Student Study Guide* to accompany Hughes-Hallett *et al.* (Wiley, 2005) 347 pp.

Ryan, Mark. *Calculus Workbook for Dummies* (Wiley Publishing, 2005) 282 pp.

St. Andre, Richard. *Study Guide for Stewart's Multivariable Calculus Fifth Edition* (Thomson, 2003) 312 pp.
From St. Andre's preface on page iii: "This *Study Guide* is designed to supplement the multivariable calculus chapters of *Calculus*, 5th edition, by James Stewart. It may also be used with *Calculus, Early Transcendentals*, 5th edition and *Multivariable Calculus*, 5th edition." I cite these details because St. Andre has also written a *Study Guide* for Stewart's *Single Variable Calculus*. That's an entirely different animal, though easily confused with the one that I reference. The 'wrong one' contains 477 pages; the 'right' one for our purpose

contains 312 pages. Meanwhile, on the Stewart side, note that I reference the *4th* edition of *Calculus*, not the 5th edition as expected, so we have a superficial disparity there (although the two editions run very close to one another, I happen to know). This potentially confusing 4th edition reference is my other reason for going into such detail about the St. Andre titles. His study guides are excellent, by the way.

Salas, Saturnino L. and Einar Hille. *Calculus: One and Several Variables* (Wiley, 1990, Sixth Edition) 1094 pp. [Comment: This one goes to a *Ninth* Edition...]

Schey, H.M. *div, grad, curl, and all that: an informal text on vector calculus* (W.W. Norton, Fourth Edition 2005 [1973]) 163 pp.

Schwartz, Lorraine. 'On Bayes procedures' in *Probability Theory and Related Fields (Historical Archive)* (Springer-Verlag GmbH 1965) 4: 1, pp. 10-26. Rani Schwartz's posthumous publication on Bayesian statistics has no direct bearing on the text. I list it only because it is 'all that remains' of a wonderful quiet spirit who once graced the streets of Berkeley, California. Only a few years after completing her doctorate there, she passed away under mysterious circumstances in Vancouver, B.C. This book is dedicated to Rani (see page iii) and to her colleagues in the statistics department at U.C. Berkeley, several of whom were frequent guests in my mother Harriett's home during the late 1950s, when I was growing up.

Simmons, George F. *Precalculus Mathematics in a Nutshell: Geometry, Algebra, Trigonometry* (Barnes & Noble, 1987) 119 pp.

Spivak, Michael. *Calculus on Manifolds: A Modern Approach to Classical Theorems of Advanced Calculus* (The Benjamin/Cummings Publishing Company, 1965) xii + 146 pp.

Stewart, James. *Calculus* (Brooks/Cole, 1999, Fourth Edition) 1185 pp.

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Wood, Beth. "Math 220 Supplemental Notes 31: Green's Theorem in the Plane" http://faculty.eicc.edu/bwood/ma220supplemental/supplemental31.htm accessed 07/05/10, 7 pp.

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Acknowledgements

Because of some car trouble, my daughter needed a ride to her chemistry class at Century College that day (while still enrolled as a senior at Mounds View High School). *Because* I was there on the Century College campus, she had an opportunity to say, "You should come upstairs to look at the chemistry lab. I think you'd like it." *Because* I poked my head into the chemistry lab, I was taken back for an instant to the bunsen burner and glassware one of my various stepfathers had bought me in Berkeley, forty-seven years earlier. And something clicked. I soon enrolled in a chemistry class. And *because* I was now in the chemistry milieu, I felt the need to take calculus... In short, thanks go first to Lady Luck, whose hand in the world is never to be underestimated (see Mlodinow, 2008).

A big thank you also to Beth Wood, whose "Math 220 Supplemental Notes 31: Green's Theorem in the Plane" I chanced upon on the Web one day. That document provides the foundation stone for **Chapter VII**.

Thank you to Fred Wahlquist for reminding me to include the cycloid and to at least mention its relatives the brachistochrone, tautochrone and trochoid, without which the chapter on '**Curves!**' could hardly have lived up to its name.

Thanks to another Medtronic colleague, Peter Stucki, for his *Anna Karenina* inversion, recalled just for fun in my prelude to Figures **31** and **32**.

More recently, thanks are due to Jane Lanctot for proofreading the math and offering other feedback on the presentation of ideas.

Graphics credits:

The solid ball and wire-frame sphere in Figure **2** were imported from Google Images. Figures **38**-**40** are based loosely on a lecture by Gerry Naughton, my calculus teacher at Century College. Thanks also to Mihail Cocos, my vector calculus teacher at University of Minnesota, for his many fine 'minimalist' examples dashed out on the white board, such as the one used as centerpiece for **Multiple Integrals** (section beginning on page **71**).

Other acknowledgements: see also **Sources/Influences** on page **10**.

Conal Boyce
St. Paul, Minnesota
January 2011

Errata

Comments/corrections welcome at conalboyce@gmail.com. My intention is to post revised pdf versions of the book from time to time at conalboyce.com.

Punctuation and Other Typographic Issues

I use quotations marks as in technical linguistics literature, which means that a single quote often appears just before a 'period dot'. Like that. To an English major, this will look wrong. So be it. It is a style that is gradually gaining ground, and it is the only style that looks right (both logically *and* aesthetically) to many of us. Double quotes I use sparingly, for special purposes.

While working as a technical editor at Medtronic (1996-2006) I became familiar with equation editors. I've never been especially impressed by the concept: steep learning curve plus acquiescence to certain canned, untweakable features. I did all the equations in this book by hand so that I could exercise pixel-level control over their components. Occasionally this avoidance of an equation editor may have resulted in an amateurish or nonstandard look but in general I believe it panned out, providing more pleasant-looking equations. One conventional practice that I reject out of hand is italicization of θ and ϕ to look like this: $\theta\ \phi$. It may be 'normal' but it looks hideous and unreadable and it adds no information (as when *a* is italicized, e.g.) After all, we expect θ to be nothing *but* a trig variable, so there is no point italicizing it to convey the message '*this is a variable*'. In this practice (rejection of italicization for θ and ϕ) I see that I have some respectable company, at least: Schey (p. 30 and passim).

For similar reasons, encouraged perhaps by a natural stinginess about buying third-party add-on packages, I plotted and drew all the curves myself. To the practiced eye, some of them will lack a certain kind of machine precision and polish. On the plus side, this approach gave me total control over their positioning, cropping and labeling. (Once I even had occasion to tweak a curve in an incorrect direction for a special purpose; the tweak is documented on page **233**, in note **14**. Using software-generated curves, that modification would not have been possible, nor would the idea itself have presented itself.)

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